Multigranulation rough sets: from partition to covering ⋆

Guoping Lin [a,b], Jiye Liang [a,∗], Yuhua Qian [a]

[a] *Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, School of Computer and Information Technology, Shanxi University, Taiyuan, 030006, Shanxi, China*

[b] *Department of Mathematics and Information Science, Zhangzhou Normal University, Zhangzhou, 363000, Fujian, China*

Abstract

The classical multigranulation rough set (MGRS) theory offers a formal theoretical framework for solving the complex problem under multigranulation environment. However, it is noticeable that MGRS theory cannot be applied in multi-source information systems with a covering environment in the real world. To address this issue, we firstly present in this paper three types of covering based multigranulation rough sets, in which set approximations are defined by different covering approximation operators. Then, by using two different approximation strategies, i.e., seeking common reserving difference and seeking common rejecting difference, two kinds of covering based multigranulation rough set are presented, namely, a covering based optimistic multigranulation rough set and a covering based pessimistic multigranulation rough sets. Finally, we develop some properties and several uncertainty measures of the covering based multigranulation rough sets. These results will enrich the MGRS theory and enlarge its application scope.

Key words: Covering, Multigranulation, Granular computing, Rough sets

1. Introduction

Rough set theory, proposed by Pawlak [30, 31], is a well-established mechanism for dealing with vagueness and uncertainty in data analysis. It is an efficient method em-

⋆ This is an extended version of the paper published at 2011 International Conference on Intelligent Computing, August 11-14, 2011, Zhengzhou, China.

∗ Corresponding author: ljy@sxu.edu.cn
Email addresses: `guoplin@163.com` (Guoping Lin), `jinchengqyh@126.com` (Yuhua Qian).

ployed in many areas: feature selection [6, 12, 13, 15, 19, 45], knowledge reduction [17, 20-23, 35], rule extraction [1, 46], uncertainty reasoning [9, 33], granular computing [3, 16, 24, 32, 50, 52], and others [5, 7, 8].

Rough set theory is originally constructed on the basis of an indiscernibility relation (or an equivalence relation) or a partition of the universe. However, it is restrictive for many real-world applications. To overcome this limitation, there are two main methods to generalize the classical rough sets. One method is to extend the equivalence relation to other binary relations, such as similarity relation, tolerance relation, and dominance relation [14, 42, 42, 47, 53]. The other important method is to replace a partition of the universe with a covering [2, 4, 11, 26, 27, 34, 41, 54-60]. In 1983, Zakowski [55] has first employed the covering of a universe for establishing a covering based generalized rough set. Since then, many researchers have proposed a great number of diversity upper and lower approximation operators and studied them extensively [2, 4, 11, 26, 27, 34, 41, 54, 56-60]. For example, Yao [54] investigated approximation operators by using coverings produced by the predecessor and/or successor neighborhoods of serial or inverse serial binary relations. Zhu et al. [56-60] systematically studied six types of approximation operators and investigated their properties and relationships of them. Particularly, Yao [54] studied a unified framework and a more systematic formulation of covering based rough sets from three aspects: the element, the granule, and the subsystem. In fact, the existing approximation operators have either dual property or non-dual property. Under the covering application background of rough sets, Chen et al. [4] presented a new covering to construct the upper and lower approximations of an arbitrary set. Covering based generalized rough sets are important improvements among these extensions, which can handle more complex practical problems. And they have obtained much attention in many domains including machine learning and uncertainty reasoning. Actually, in the view of granular computing [52], either a partition or a covering of the universe can be considered as a granular space.

From the above, we can see that set approximations in the above rough sets are described only by a single binary relation (a single granulation [52]) or a single covering (or a single covering granulation) on a given universe, which cannot be applied in some practical multigranulation backgrounds [36, 37]. Qian et al. [36] first took multiple binary relations into account and proposed multigranulation rough sets, in which a target concept was described by multiple binary relations on a universe according to a user's different requirements. Up to now, many extensions of MGRS have been proposed. For example, Liu et al. [28, 29] proposed covering fuzzy rough set based multigranulation rough sets. Xu et al. [48] investigated another generalized version, called variable precision multigranulation rough sets. Yang et al. [51] proposed a multigranulation rough set based on a fuzzy binary relation. Lin et al. [25] investigated neighborhood-based multigranulation rough sets, which can be used to deal with data sets with hybrid attributes. She et al. [44] explored topological structures of multigranulation rough sets, which further enriches the theory of MGRS. It is deserved to mention that Liang et al.[15] proposed an efficient feature selection algorithm for large-scale data sets from the perspective of multiple granulations, which has shown an important implication of MGRS theory. Accordingly, MGRS theory has displayed its advantages in knowledge discovery from large-scale data sets. In fact, in a Pawlak's approximation space, each object can be classified into a certain concept as shown in Figure 1. However, in real-world applications, such as a multi-source covering information system [10] and computing with words, different subsets of the universe usually overlap, as shown in Figure 2, in which these basic information granules form a covering of the objects, rather than a Pawlak's approximation space. It is difficult for the classical MGRS theory to deal with this issue. To address this issue, it is necessary to generalize the classical MGRS to covering based multigranulation rough sets for enriching its application domains.

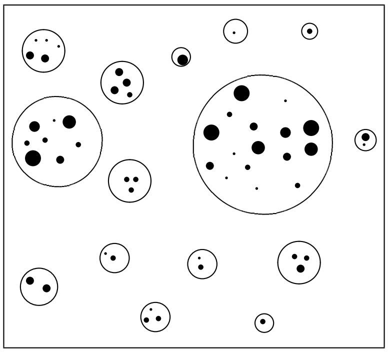

Figure 1

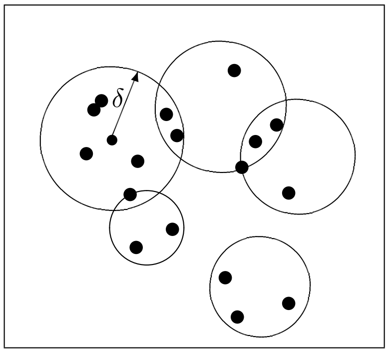

Figure 2

In this paper, we introduce covering into the multigranulation environment and present covering based optimistic and pessimistic multigranulation rough sets.

Additionally, lots of researchers suggested some possible applications of the uncertainty measures in the fields of pattern recognition and image analysis in the literature [9, 18, 39-41, 49]. The concept of entropy was originally introduced by Shannon in [40], which is a very useful mechanism for characterizing information content in various modes. It has been applied in many diverse fields. Furthermore, Shannon entropy and its variants were adopted for rough set theory in the literature [9, 18, 39, 41, 49]. Similarly, in this paper, in order to make wide applications of the covering based multigranulation rough set theory, we propose several uncertainty measures for covering based multigranulation rough sets, including degree of rough membership, approximation measure, and rough entropy.

The main objective of this paper is to establish three types of rough sets based on multiple coverings by using different approximation strategies due to the practical different applied backgrounds. The rest of this paper is organized as follows. Some basic concepts of classical multigranulation rough sets are briefly reviewed in Section 2. In Section 3, three types of covering based optimistic and pessimistic multigranulation rough sets are constructed and some of their important properties are investigated. In Section 4, several uncertainty measures for covering based multigranulation rough sets are presented, such as degree of rough membership, approximation measure, and rough entropy. We then conclude the paper with a summary and direction for the further research in the last section.

2. Preliminaries

In this section, we review some basic concepts of covering based rough sets and multigranulation rough sets [4, 36, 55, 56]. Throughout this paper, we suppose the universe of discourse U is a finite non-empty set.

2.1. *Covering based rough sets*

Let U be a finite non-empty set of objects and $\mathcal{C}$ a family of subsets of U. If no subset in $\mathcal{C}$ is empty and $\bigcup \mathcal{C} = U$, $\mathcal{C}$ is called a covering of U. Then the ordered pair $< U, \mathcal{C} >$ is called a covering approximation space.

Definition 2.1 [2]. Let $(U, \mathcal{C})$ be a covering approximation space. For $x \in U$, the minimal description of x is defined as

$$Md(x) = \{K \in \mathcal{C} \mid (x \in K) \wedge (x \in S \in \mathcal{C} \wedge S \subseteq K \Longrightarrow S = K)\}.$$

If $|Md(x)| = 1$, x is called a representative element of K.

Definition 2.2 [56]. Let $(U, \mathcal{C})$ be a a covering approximation space. For $x \in U$, the neighborhood of x is defined as

$$N(x) = \cap\{K \in \mathcal{C} \mid x \in K\}.$$

There are dozens of approximation operators for covering based rough sets to deal with the diversity formed by covering data. However, in this paper, inspired by Yao's study [54], we only list three pairs of operators to illustrate the idea of the forthcoming covering based multigranulation rough sets.

Definition 2.3 [54, 56, 59]. Let $(U, \mathcal{C})$ be a covering approximation space. For each $i \in \{1, 2, 3\}$, $\underline{\mathcal{C}_i}$ and $\overline{\mathcal{C}_i}$ called the i-th lower covering approximation operator and the i-th upper covering approximation operator on $(U, \mathcal{C})$ are defined as follows.

(1) $\underline{\mathcal{C}_1}(X) = \cup\{K \in \mathcal{C} \mid K \subseteq X\}$, (Granule based Definition)
$\overline{\mathcal{C}_1}(X) = \sim \underline{\mathcal{C}_1}(\sim X)$
$= \{x | x \in U, \forall K \in \mathcal{C} | x \in K \Rightarrow K \cap A \neq \emptyset\}$.

(2) $\underline{\mathcal{C}_2}(X) = \{x \in U \mid N(x) \subseteq X\}$, (Element based Definition)
$\overline{\mathcal{C}_2}(X) = \{x \in U \mid N(x) \cap X \neq \emptyset\}$.

(3) $\underline{\mathcal{C}_3}(X) = \cup\{K \in \mathcal{C} \mid K \subseteq X\}$, (Granule based Definition)
$\overline{\mathcal{C}_3}(X) = \cup\{K \in \mathcal{C} \mid K \cap X \neq \emptyset\}$.

From the above, we call $(\underline{\mathcal{C}_i}(X), \overline{\mathcal{C}_i}(X))$, $i = \{1, 2, 3\}$, single covering based rough sets. The pairs of approximation operators (1), (2) and (3) can be found in the literatures[54], [56], [59], respectively.

Definition 2.4. Let U be a universe of discourse, $\mathcal{C}_1 = \{K_{11}, K_{12}, \cdots, K_{1|C_1|}\}$, $\mathcal{C}_2 = \{K_{21}, K_{22}, \cdots, K_{2|\mathcal{C}_2|}\}$ two different coverings of U. An intersection operation between $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined as follows:

$$\mathcal{C}_1 \cap \mathcal{C}_2 = \{K_{1i} \cap K_{2j} \mid K_{1i} \cap K_{2j} \neq \emptyset, K_{1i} \in \mathcal{C}_1, K_{2j} \in \mathcal{C}_2, 1 \leq i \leq |\mathcal{C}_1|, 1 \leq j \leq |\mathcal{C}_2|\}.$$

where $|\mathcal{C}_i|$ represents the cardinality of $\mathcal{C}_i$. In what follows, we denote $t_i = |\mathcal{C}_i|$ for simplicity.

Proposition 2.1 [4]. Let $\mathcal{C} = \{K_1, K_2, \cdots, K_t\}$ be a covering of U. For every $x \in U$, suppose $\mathcal{C}_x = \bigcap\{K_i \mid K_i \in \mathcal{C}, x \in K_i\}$. Then $Cov(\mathcal{C}) = \{\mathcal{C}_x | x \in U\}$ is a covering of U.

Proposition 2.2 [4]. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U, where $C_i = \{K_{i1}, K_{i2}, \cdots, K_{it_i}\}$, $i = 1, 2, \cdots, m$. For $X \subseteq U$, suppose $\Omega_x = \bigcap\{(K_{ij})_x \mid (K_{ij})_x \in Cov(\mathcal{C}_i), i \in \{1, 2, \cdots, m\}, j = 1, 2, \cdots, t_i\}$. Then, $Cov(\Omega) = \{\Omega_x \mid x \in U\}$ is a covering of U. Throughout this paper, we use $(K_{ij})_x$ to represent a set including x in $Cov(\Omega)$.

Definition 2.5 [4]. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U. For $X \subseteq U$, the lower and upper approximations of X with respect to $Cov(\Omega)$ are defined as follows:

$$\underline{\Omega}(X) = \bigcup\{\Omega_x \mid \Omega_x \subseteq X\},$$

$$\overline{\Omega}(X) = \bigcup\{\Omega_x \mid \Omega_x \cap X \neq \emptyset\}.$$

Here, we use an example to illustrate the above definitions and propositions.

Example 2.1. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be a universe. $\mathcal{C}_1 = \{C_{11} = \{x_1, x_2, x_4, x_5\}, C_{12} = \{x_2, x_5\}, C_{13} = \{x_3, x_5\}\}$ and $\mathcal{C}_2 = \{C_{21} = \{x_1, x_2, x_3\}, C_{22} = \{x_4, x_5\}, C_{23} = \{x_2, x_4\}\}$ are two coverings of U. For the covering C_1, by Proposition 2.1, we have that $C_{1_{x_1}} = C_{11} = \{x_1, x_2, x_4, x_5\}$, $C_{1_{x_2}} = C_{11} \cap C_{12} = \{x_2, x_5\}$, $C_{1_{x_3}} = C_{13} = \{x_3, x_5\}$, $C_{1_{x_4}} = C_{11} = \{x_1, x_2, x_4, x_5\}$ and $C_{1_{x_5}} = C_{11} \cap C_{12} \cap C_{13} = \{x_5\}$. Obviously, $Cov(\mathcal{C}_1) = \{C_{1_{x_1}}, C_{1_{x_2}}, C_{1_{x_3}}, C_{1_{x_4}}, C_{1_{x_5}}\}$ also forms a covering of U. Similarly, for the covering $\mathcal{C}_2$, we have that $C_{2_{x_1}} = C_{22} = \{x_1, x_2, x_3\}$, $C_{2x_2} = C_{21} \cap C_{23} = \{x_2\}$, $C_{2_{x_3}} = C_{22} = \{x_1, x_2, x_3\}$, $C_{2_{x_4}} = C_{22} \cap C_{23} = \{x_4\}$, and $C_{2_{x_5}} = C_{22} = \{x_4, x_5\}$. Obviously, $Cov(\mathcal{C}_2) = \{C_{2_{x_1}}, C_{2_{x_2}}, C_{2_{x_3}}, C_{2_{x_4}}, C_{2_{x_5}}\}$ also forms a covering of U.

By Proposition 2.2, we have that $\Omega_{x_1} = C_{1_{x_1}} \cap C_{2_{x_1}} \cap C_{2_{x_3}} = \{x_1, x_2\}$, $\Omega_{x_2} = \{x_2\}$, $\Omega_{x_3} = \{x_2\}$, $\Omega_{x_4} = \{x_4\}$, and $\Omega_{x_5} = \{x_5\}$. Obviously, $Cov(\Omega) = \{\Omega_{x_1}, \Omega_{x_2}, \Omega_{x_3}, \Omega_{x_4}, \Omega_{x_5}\}$ also forms a covering of U.

Suppose that $X = \{x_1, x_3, x_4\} \subseteq U$. According to Definition 2.5, we have that $\underline{\Omega}(X) = \{x_2, x_3\}$ and $\overline{\Omega}(X) = \{x_1, x_2, x_3, x_4\}$.

2.2. *Multigranulation rough sets*

According to two different approximation strategies, Qian et. al [36, 38] developed two different multigranulation rough sets (MGRS) including optimistic and pessimistic ones.

Definition 2.6. Let $S = (U, AT, f)$ be a complete information system, $A_1, A_2, \cdots, A_m \subseteq AT$, and $X \subseteq U$. The optimistic lower and upper approximations of X with respect to $A_1, A_2, \cdots, A_m$ are denoted by $\underline{\sum_{i=1}^{m} A_i}^{O} X$ and $\overline{\sum_{i=1}^{m} A_i}^{O} X$, respectively, where

$$\underline{\sum_{i=1}^{m} A_i}^{O}(X) = \{x \in U \mid [x]_{A_1} \subseteq X \vee [x]_{A_2} \subseteq X \vee \cdots \vee [x]_{A_m} \subseteq X\},$$

$$\overline{\sum_{i=1}^{m} A_i}^{O}(X) = \sim \underline{\sum_{i=1}^{m} A_i}^{O}(\sim X).$$

Then $(\underline{\sum_{i=1}^m A_i}^O(X), \overline{\sum_{i=1}^m A_i}^O(X))$ is called the classical optimistic MGRS [36].

Let $\emptyset$ be an empty set and $\sim X$ the complement of X in U. We have the following properties of optimistic multigranulation rough sets [36].

(1OML) $\underline{\sum_{i=1}^m A_i}^O(U) = U$ (Co-normality)

(1OMH) $\overline{\sum_{i=1}^m A_i}^O(U) = U$ (Co-normality)

(2OML) $\underline{\sum_{i=1}^m A_i}^O(\emptyset) = \emptyset$ (Normality)

(2OMH) $\overline{\sum_{i=1}^m A_i}^O(\emptyset) = \emptyset$ (Normality)

(3OML) $\underline{\sum_{i=1}^m A_i}^O(X) \subseteq X$ (Contraction)

(3OMH) $X \subseteq \overline{\sum_{i=1}^m A_i}^O(X)$ (Extension)

(4OML) $\underline{\sum_{i=1}^m A_i}(\bigcap_{j=1}^n X_j) \subseteq \bigcap_{j=1}^n(\underline{\sum_{j=1}^n A_i}(X_j))$ (Implication)

(4OMH) $\overline{\sum_{i=1}^m A_i}(\bigcup_{j=1}^n X_j) \supseteq \bigcup_{j=1}^n(\overline{\sum_{j=1}^n A_i}(X_j))$ (Implication)

(5OML) $\underline{\sum_{i=1}^m A_i}(\bigcup_{j=1}^n X_j) \supseteq \bigcup_{j=1}^n(\underline{\sum_{j=1}^n A_i}(X_j))$ (Implication)

(5OMH) $\overline{\sum_{i=1}^m A_i}(\bigcap_{j=1}^n X_j) \subseteq \bigcap_{j=1}^n(\overline{\sum_{j=1}^n A_i}(X_j))$ (Implication)

(6OML) $\underline{\sum_{i=1}^m A_i}^O(\underline{\sum_{i=1}^m A_i}^O(X)) = \underline{\sum_{i=1}^m A_i}^O(X)$ (Idempotency)

(6OMH) $\overline{\sum_{i=1}^m A_i}^O(\overline{\sum_{i=1}^m A_i}^O(X)) = \overline{\sum_{i=1}^m A_i}(X)$ (Idempotency)

(7OML) $\underline{\sum_{i=1}^m A_i}^O(\sim X)) = \sim \overline{\sum_{i=1}^m A_i}^O(X)$ (Duality)

(7OMH) $\overline{\sum_{i=1}^m A_i}^O(\sim X) = \sim \underline{\sum_{i=1}^m A_i}^O(X)$ (Duality)

(8OML) $X \subseteq Y \Rightarrow \underline{\sum_{i=1}^m A_i}^O(X) \subseteq \underline{\sum_{i=1}^m A_i}^O(Y)$ (Monotone)

(8OMH) $X \subseteq Y \Rightarrow \overline{\sum_{i=1}^m A_i}^O(X) \subseteq \overline{\sum_{i=1}^m A_i}^O(Y)$ (Monotone)

(9OML) $\forall K \in U/A_i, i \in \{1, 2, \cdots, m\}, \underline{\sum_{i=1}^m A_i}^O(K) = K$ (Granularity)

(9OMH) $\forall K \in U/A_i, i \in \{1, 2, \cdots, m\}, \overline{\sum_{i=1}^m A_i}^O(K) = K$ (Granularity)

(10OML) $\underline{\sum_{i=1}^m A_i}^O(X) = \bigcup_{i=1}^m(\underline{A_i}(X))$ (Relation based Addition)

(10OMH) $\overline{\sum_{i=1}^m A_i}^O(X) = \bigcap_{i=1}^m(\overline{A_i}(X))$ (Relation based Multiplication)

In addition, the definition of the classical pessimistic MGRS [38] is defined as follows:

$$\underline{\sum_{i=1}^m A_i}^P(X) = \{x \in U \mid [x]_{A_1} \subseteq X \wedge [x]_{A_2} \subseteq X \wedge \cdots \wedge [x]_{A_m} \subseteq X\},$$

$$\overline{\sum_{i=1}^m A_i}^P(X) = \sim \underline{\sum_{i=1}^m A_i}^P(\sim X).$$

Let $\emptyset$ be an empty set and $\sim X$ the complement of X in U. The pessimistic multigranulation rough sets have the following properties [38].

(1PML) $\underline{\sum_{i=1}^{m} A_i}^{P}(U) = U$ (Co-normality)

(1PMH) $\overline{\sum_{i=1}^{m} A_i}^{P}(U) = U$ (Co-normality)

(2PML) $\underline{\sum_{i=1}^{m} A_i}^{P}(\emptyset) = \emptyset$ (Normality)

(2PMH) $\overline{\sum_{i=1}^{m} A_i}^{P}(\emptyset) = \emptyset$ (Normality)

(3PML) $\underline{\sum_{i=1}^{m} A_i}^{P}(X) \subseteq X$ (Contraction)

(3PMH) $X \subseteq \overline{\sum_{i=1}^{m} A_i}^{P}(X)$ (Extension)

(4PML) $\underline{\sum_{i=1}^{m} A_i}^{P}(\bigcap_{j=1}^{n} X_j) = \bigcap_{j=1}^{n}(\underline{\sum_{j=1}^{n} A_i}^{P}(X_j))$ (Implication)

(4PMH) $\overline{\sum_{i=1}^{m} A_i}^{P}(\bigcup_{j=1}^{n} X_j) = \bigcup_{j=1}^{n}(\overline{\sum_{j=1}^{n} A_i}^{P}(X_j))$ (Implication)

(5PML) $\underline{\sum_{i=1}^{m} A_i}^{P}(\bigcup_{j=1}^{n} X_j) \supseteq \bigcup_{j=1}^{n}(\underline{\sum_{j=1}^{n} A_i}^{P}(X_j))$ (Coarse Implication)

(5PMH) $\overline{\sum_{i=1}^{m} A_i}^{P}(\bigcap_{j=1}^{n} X_j) \subseteq \bigcap_{j=1}^{n}(\overline{\sum_{j=1}^{n} A_i}^{P}(X_j))$ (Fine Implication)

(6PMH) $\overline{\sum_{i=1}^{m} A_i}^{P}(\overline{\sum_{i=1}^{m} A_i}^{P}(X)) = \overline{\sum_{i=1}^{m} A_i}^{P}(X)$ (Idempotency)

(7PML) $\underline{\sum_{i=1}^{m} A_i}^{P}(\sim X)) =\sim \overline{\sum_{i=1}^{m} A_i}^{P}(X)$ (Duality)

(7PMH) $\overline{\sum_{i=1}^{m} A_i}^{P}(\sim X) =\sim \underline{\sum_{i=1}^{m} A_i}^{P}(X)$ (Duality)

(8PML) $X \subseteq Y \Rightarrow \underline{\sum_{i=1}^{m} A_i}^{P}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^{P}(Y)$ (Monotone)

(8PMH) $X \subseteq Y \Rightarrow \overline{\sum_{i=1}^{m} A_i}^{P}(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^{P}(Y)$ (Monotone)

(9PML) $\forall K \in U/A_i, i \in \{1, 2, \cdots, m\}, \underline{\sum_{i=1}^{m} A_i}^{P}(K) = K$ (Granularity)

(9PMH) $\forall K \in U/A_i, i \in \{1, 2, \cdots, m\}, \overline{\sum_{i=1}^{m} A_i}^{P}(K) = K$ (Granularity)

(10PML) $\underline{\sum_{i=1}^{m} A_i}^{P}(X) = \bigcap_{i=1}^{m}(\underline{A_i}(X))$ (Relation based Addition)

(10PMH) $\overline{\sum_{i=1}^{m} A_i}^{P}(X) = \bigcup_{i=1}^{m}(\overline{A_i}(X))$ (Relation based Multiplication)

3. Covering based multigranulation rough sets

In the previous research work, covering based rough sets are constructed by one single covering (or a single covering granulation space) of the universe. Even though multiple coverings induced by neighborhood relations have been used in [25], they are only special ones in covering based multigranulation rough sets. Therefore, in order to enlarge the application scope of MGRS and enrich its theory, we introduce multiple coverings into covering based rough sets by the idea of MGRS. According to the first, the second, and the third pairs of the covering approximation operators in Definition 2.3 in Section 2, in this section, we correspondingly propose three types of covering based multigranulation rough sets. Furthermore, based on two different approximation strategies, we also investigate optimistic and pessimistic ones of each proposed covering based MGRS.

3.1. *The first type of covering approximation operators based multigranulation rough sets (Or the first type of CMGRS)*

3.1.1. *The first type of optimistic CMGRS*

Let U be a finite universe of discourse, $\mathcal{C}_1$ and $\mathcal{C}_2$ two different coverings of U, $K_x \subseteq U$, and $x \in K_x$. For any $K_x \in \mathcal{C}_1$, if there exists $L_x \in \mathcal{C}_2$ such that $K_x \subseteq L_x$, we call that $\mathcal{C}_1$ is uniformly finer than $\mathcal{C}_2$ (or $\mathcal{C}_2$ is uniformly coarser than $\mathcal{C}_1$), called a uniform partial relation between $\mathcal{C}_1$ and $\mathcal{C}_2$, denoted by $\mathcal{C}_1 \preceq^c \mathcal{C}_2$. If $\mathcal{C}_1 \preceq^c \mathcal{C}_2$ and $\mathcal{C}_1 \neq \mathcal{C}_2$, we say that $\mathcal{C}_1$ is strictly finer than $\mathcal{C}_2$ (or $\mathcal{C}_2$ is strictly coarser than $\mathcal{C}_1$), written as $\mathcal{C}_1 \prec^c \mathcal{C}_2$.

Especially, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two different partitions of U, K_x is a subset including x. If for any $K_x \in \mathcal{C}_1$, there exists $L_x \in \mathcal{C}_2$ such that $K_x \subseteq L_x$, we call that $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ (or $\mathcal{C}_2$ is coarser than $\mathcal{C}_1$), denoted by $\mathcal{C}_1 \preceq \mathcal{C}_2$. If $\mathcal{C}_1 \preceq \mathcal{C}_2$ and $\mathcal{C}_1 \neq \mathcal{C}_2$, we say that $\mathcal{C}_1$ is strictly finer than $\mathcal{C}_2$ (or $\mathcal{C}_2$ is strictly coarser than $\mathcal{C}_1$), written as $\mathcal{C}_1 \prec \mathcal{C}_2$.

K_x represents a subset including x throughout this paper.

Theorem 3.1. The partial relation $\preceq$ is a special case of a uniform partial relation $\preceq^c$.

Proof. If $\mathcal{C}$ is a partition instead of a covering of U, then it is obvious that the uniform partial relation $\preceq^c$ degenerates into the partial relation $\preceq$.

Example 3.1 (Continued from Example 2.1). Suppose $\mathcal{C}_1 = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_5\}\}$ and $\mathcal{C}_2 = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}\}$. Then, we have that $\mathcal{C}_1 \preceq \mathcal{C}_2$ and $\mathcal{C}_1 \preceq^c \mathcal{C}_2$.

Definition 3.1. Let (U, Ω) be a covering approximation space, $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ a family of coverings of U with $\mathcal{C}_i = \{K_{i1}, K_{i2}, \cdots, K_{it_i}\}$, and $X \subseteq U$. An optimistic lower approximation and an optimistic upper approximation of X with respect to Ω, denoted by $\underline{\sum_{i=1}^m \mathcal{C}_i}^O(X)$ and $\overline{\sum_{i=1}^m \mathcal{C}_i}^O(X)$, respectively, are defined by the following

$$\underline{\sum_{i=1}^m \mathcal{C}_i}^O(X) = \bigcup\{K_{ij} \in \mathcal{C}_i \mid \vee(K_{ij} \subseteq X), i \in \{1, 2, \cdots, m\}; j = 1, 2, \cdots, t_i\}, \quad (1)$$

$$\overline{\sum_{i=1}^m \mathcal{C}_i}^O(X) = \sim \underline{\sum_{i=1}^m \mathcal{C}_i}^O(\sim X). \quad (2)$$

And the *area of uncertainty* or *boundary region* of X relative to Ω in the covering based multigranulation rough sets is

$$Bn^O_{\sum_{i=1}^m \mathcal{C}_i}(X) = \overline{\sum_{i=1}^m \mathcal{C}_i}^O(X) \backslash \underline{\sum_{i=1}^m \mathcal{C}_i}^O(X).$$

Then, $(\underline{\sum_{i=1}^m \mathcal{C}_i}^O(X), \overline{\sum_{i=1}^m \mathcal{C}_i}^O(X))$ is called the first type of covering based optimistic multigranulation rough sets (or the first type of optimistic CMGRS, for short). For simplicity, we say $(U, \mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m)$ an optimistic multigranulation covering approximation space, denoted by $((U, \sum^O \mathcal{C}_i))$, i. e. OMCA-Space .

Remark 1. In a special case, when $i = 1$, the first type of optimistic CMGRS will degenerate into a single covering based rough set whose lower and upper approximation operations are just (1) of Definition 2.3. In addition, if $\mathcal{C}_i, i \in \{1, 2, \cdots, m\}$, is a partition on the universe U, then $(\underline{\sum_{i=1}^{m} C_i}^{O}(X), \overline{\sum_{i=1}^{m} \mathcal{C}_i}^{O}(X))$ will degenerate into the original MGRS. According to Yao's opinion [54], we say that this pair of approximation operators is defined by the granule.

Theorem 3.2. Let (U, Ω) be a covering approximation space, $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ a family of coverings of U, and $X \subseteq U$. Then,

$$\overline{\sum_{i=1}^{m} \mathcal{C}_i}^{O}(X) = \{x \in U \mid \wedge((K_{ij})_x \cap X \neq \emptyset), i \in \{1, 2, \cdots, m\}; j = 1, 2, \cdots, t_i\}, \tag{3}$$

where $C_i = \{K_{i1}, K_{i2}, \cdots, K_{it_i}\}$.

Proof. $x \in \overline{\sum_{i=1}^{m} \mathcal{C}_i}^{O}(X) \Leftrightarrow x \in \sim \underline{\sum_{i=1}^{m} \mathcal{C}_i}^{O}(\sim X)$

$$\begin{aligned} &\Leftrightarrow x \notin \underline{\sum_{i=1}^{m} \mathcal{C}_i}^{O}(\sim X) \\ &\Leftrightarrow (K_{ij})_x \nsubseteq (\sim X), i \in \{1, 2, \cdots, m\}, j = 1, 2, \cdots, t_i \\ &\Leftrightarrow \wedge((K_{ij})_x \cap X \neq \emptyset), i \in \{1, 2, \cdots, m\}, j = 1, 2, \cdots, t_i. \end{aligned}$$

By Theorem 3.2, we see that though the optimistic multigranulation upper approximation is defined by the complement of the optimistic multigranulation lower approximation, it can also be constructed by objects with non-empty intersection with the target concept in terms of each granular structure.

In order to illustrate Definition 3.1, we here continue to use the common example from the literature [4].

Example 3.2. Let us consider an evaluation problem of a credit card applicant. Suppose that $U = \{x_1, x_2, \cdots, x_9\}$ is a set of nine applicants. $E = \{$*education*, *salary*$\}$ is a set of two condition attributes. The values of attribute "*education*" are $\{$*best, better, good*$\}$. And the values of attribute"*salary*" are $\{$*high, middle, low*$\}$. We make three specialists A, B, C evaluate the attribute values for these applicants. It is possible that their evaluation results to the same attribute values may not be the same each other. The evaluation results are listed below as Table 1. In Table 1, y_i $(i = 1, 2, 3)$ denote the evaluation results given by specialists A, B, C, respectively, as well as n_i $(i = 1, 2, 3)$, where y_i means "yes" and n_i means "no".

Example 3.3 (Continued from Example 3.1). From Table 1, for the attribute "education", the specialist A gives evaluation results: the applicants x_1, x_4, x_5, and x_7 get "best", denoted by $best = \{x_1, x_4, x_5, x_7\}$, the applicants x_2 and x_8 get "better", denoted by $better = \{x_2, x_8\}$, and the applicants x_3, x_6, and x_9 get "good", denoted by $good = \{x_3, x_6, x_9\}$. In brief, we denote that
A: $\mathcal{C}_1 = \{best = \{x_1, x_4, x_5, x_7\}, better = \{x_2, x_8\}, good = \{x_3, x_6, x_9\}\}$;
Similarly, we get that
B: $\mathcal{C}_2 = \{best = \{x_1, x_2, x_4, x_5, x_7, x_8\}, better = \{x_2, x_5, x_8\}, good = \{x_3, x_5, x_6, x_9\}\}$;
C: $\mathcal{C}_3 = \{best = \{x_4, x_7\}, better = \{x_2, x_8\}, good = \{x_1, x_3, x_5, x_6, x_9\}\}$.
And for the attribute "salary", we have that
A: $\mathcal{C}_4 = \{high = \{x_1, x_2, x_3\}, middle = \{x_4, x_5, x_6, x_7, x_8\}, low = \{x_2, x_5, x_9\}\}$;
B: $\mathcal{C}_5 = \{high = \{x_1, x_2, x_3\}, middle = \{x_4, x_5, x_6, x_7, x_8\}, low = \{x_7, x_8, x_9\}\}$;
C: $\mathcal{C}_6 = \{high = \{x_1, x_2, x_3\}, middle = \{x_4, x_5, x_6, x_8\}, low = \{x_7, x_9\}\}$.

Table 1
An evaluation information system

U \ A	Education									Salary								
Attribute value	Best			Better			Good			High			Middle			Low		
	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C
x_1	y_1	y_2	n_3	n_1	n_2	n_3	n_1	n_2	y_3	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3
x_2	n_1	y_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	y_1	n_2	n_3
x_3	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3
x_4	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3
x_5	y_1	y_2	n_3	n_1	y_2	n_3	n_1	y_2	y_3	n_1	n_2	n_3	y_1	y_2	y_3	y_1	n_2	n_3
x_6	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3
x_7	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	n_3	n_1	y_2	y_3
x_8	n_1	y_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	y_2	n_3
x_9	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3

Therefore, $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$ are six coverings of U. We choose randomly two coverings $\mathcal{C}_2 = \{\{x_1, x_2, x_4, x_5, x_7, x_8\}, \{x_2, x_5, x_8\}, \{x_3, x_5, x_6, x_9\}\}$ and $\mathcal{C}_5 = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6, x_7, x_8\}, \{x_7, x_8, x_9\}\}$ from them. For a target concept $X = \{x_1, x_2, x_5, x_8\} \subseteq U$, by Definition 3.1, one has that $\underline{\mathcal{C}_1 + \mathcal{C}_2}^O(X) = \{x_2, x_5, x_8\} \cup \emptyset = \{x_2, x_5, x_8\}$ and $\overline{\mathcal{C}_1 + \mathcal{C}_2}^O(X) = U$. Then, by Definition 2.4, we get a new covering of the universe, i. e., $\mathcal{C}_1 \cap \mathcal{C}_2 = \{\{x_1, x_2\}, \{x_2\}, \{x_3\}, \{x_4, x_5, x_7, x_8\}, \{x_5, x_8\}, \{x_5, x_6\}, \{x_6, x_9\}, \{x_8\}, \{x_9\}, \{x_7, x_8\}\}$. Then, $\underline{\mathcal{C}_1 \cap \mathcal{C}_2}(X) = \{x_1, x_2, x_5, x_8\}$, $\overline{\mathcal{C}_1 \cap \mathcal{C}_2}(X) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8\}$. Hence, we have that $(\underline{\mathcal{C}_1 + \mathcal{C}_2})^O(X) \subseteq \underline{\mathcal{C}_1 \cap \mathcal{C}_2}(X)$ and $(\overline{\mathcal{C}_1 + \mathcal{C}_2})^O(X) \supseteq \overline{\mathcal{C}_1 \cap \mathcal{C}_2}(X)$.

As a result of this example, we see that the optimistic lower approximation of X induced by $\mathcal{C}_1 + \mathcal{C}_2$ is not more than that induced by $\mathcal{C}_1 \cap \mathcal{C}_2$. Then we have the following propositions.

Proposition 3.1. Let (U, Ω) be a covering approximation space, $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ a family of coverings of U, and $X \subseteq U$. Then,

(1) $\underline{\sum_{i=1}^m \mathcal{C}_i}^O(X) \subseteq \underline{\bigcap_{i=1}^m \mathcal{C}_i}(X)$,

(2) $\overline{\sum_{i=1}^m \mathcal{C}_i}^O(X) \supseteq \overline{\bigcap_{i=1}^m \mathcal{C}_i}(X)$.

Proof. (1) For any $x \in \underline{\sum_{i=1}^m \mathcal{C}_i}^O(X)$, by Definition 3.1, it follows that there must exist $(K_{ij})_x \in \mathcal{C}_i, i \in \{1, 2, \cdots, m\}, j = \{1, 2, \cdots, t_i\}$ such that $x \in (K_{ij})_x$. Here, we use $(K_{ij})_x$ to denote a set which includes x. By Definition 2.4, we know that $\mathcal{C}_x \subseteq (K_{ij})_x$. Obviously, if $\mathcal{C}_x \subseteq X$, $(K_{ij})_x$ is not always included in X. Conversely, it holds. Hence, $\underline{\sum_{i=1}^m \mathcal{C}_i}^O(X) \subseteq \underline{\bigcap_{i=1}^m \mathcal{C}_i}(X)$.

(2) For any $x \in \overline{\bigcap_{i=1}^m \mathcal{C}_i}(X)$, there exists $\mathcal{C}_x$ such that $x \in \mathcal{C}_x$ and $\mathcal{C}_x \cap X \neq \emptyset$. But $\mathcal{C}_x \subseteq (K_{ij})_x$ $(j = 1, 2, \cdots, t)$. Obviously, $(K_{ij})_x \cap X \neq \emptyset$, i. e., $x \in \overline{\sum_{i=1}^m \mathcal{C}_i}^O(X)$. Therefore, $\overline{\sum_{i=1}^m \mathcal{C}_i}^O(X) \supseteq \overline{\bigcap_{i=1}^m \mathcal{C}_i}(X)$.

Proposition 3.2. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U, and $X \subseteq U$. Then, the following properties hold

(1) $\underline{\sum_{i=1}^m \mathcal{C}_i}^O U = \overline{\sum_{i=1}^m \mathcal{C}_i}^O U = U$,

(2) $\underline{\sum_{i=1}^m \mathcal{C}_i}^O \emptyset = \overline{\sum_{i=1}^m \mathcal{C}_i}^O \emptyset = \emptyset$,

(3) $\underline{\sum_{i=1}^m \mathcal{C}_i}^O (X) \subseteq X \subseteq \overline{\sum_{i=1}^m \mathcal{C}_i}^O (X)$,

(4) $\underline{\sum_{i=1}^m \mathcal{C}_i}^O (X \cap Y) \subseteq \underline{\sum_{i=1}^m \mathcal{C}_i}^O (X) \cap \underline{\sum_{i=1}^m \mathcal{C}_i}^O (Y)$,

(5) $\overline{\sum_{i=1}^m \mathcal{C}_i}^O (X \cup Y) \supseteq \overline{\sum_{i=1}^m \mathcal{C}_i}^O (X) \cup \overline{\sum_{i=1}^m \mathcal{C}_i}^O (Y)$,

(6) $\underline{\sum_{i=1}^m \mathcal{C}_i}^O (X \cup Y) \supseteq \underline{\sum_{i=1}^m \mathcal{C}_i}^O (X) \cup \underline{\sum_{i=1}^m \mathcal{C}_i}^O (Y)$,

(7) $\overline{\sum_{i=1}^m \mathcal{C}_i}^O (X \cap Y) \subseteq \overline{\sum_{i=1}^m \mathcal{C}_i}^O (X) \cap \overline{\sum_{i=1}^m \mathcal{C}_i}^O (Y)$,

(8) $\underline{\underline{\sum_{i=1}^m \mathcal{C}_i}}^O (X) = \underline{\sum_{i=1}^m \mathcal{C}_i}^O (X)$,

(9) $\underline{\sum_{i=1}^m \mathcal{C}_i}^O (X) = \sim \overline{\sum_{i=1}^m \mathcal{C}_i}^O (\sim X)$,

(10) $\overline{\sum_{i=1}^m \mathcal{C}_i}^O (X) = \sim \underline{\sum_{i=1}^m \mathcal{C}_i}^O (\sim X)$,

(11) $X \subseteq Y \Rightarrow \underline{\sum_{i=1}^m \mathcal{C}_i}^O (X) \subseteq \underline{\sum_{i=1}^m \mathcal{C}_i}^O (Y)$,

(12) $X \subseteq Y \Rightarrow \overline{\sum_{i=1}^m \mathcal{C}_i}^O (X) \subseteq \overline{\sum_{i=1}^m \mathcal{C}_i}^O (Y)$.

Proof. They can be easily proved by Definition 3.1.

However, $\overline{\overline{\sum_{i=1}^m \mathcal{C}_i}}^O (X) = \overline{\sum_{i=1}^m \mathcal{C}_i}^O (X)$ may not hold.

For example, let $U = \{x_1, x_2, x_3, x_4\}$ be a universe, $\mathcal{C}_1 = \{\{x_1, x_3\}, \{x_1\}, \{x_2\}, \{x_3, x_4\}\}$ and $\mathcal{C}_2 = \{\{x_2, x_3\}, \{x_2\}, \{x_3, x_4\}, \{x_1\}\}$ two coverings of U. For $X = \{x_1, x_2\}$, $\overline{\mathcal{C}_1 + \mathcal{C}_2}^O (X) = \{x_1, x_2, x_3\}$ and $\overline{\overline{\mathcal{C}_1 + \mathcal{C}_2}}^O (X) = \{x_1, x_2, x_3, x_4\}$. Hence, $\overline{\overline{\sum_{i=1}^m \mathcal{C}_i}}^O (X) \neq \overline{\sum_{i=1}^m \mathcal{C}_i}^O (X)$.

This example shows that a distinction between the classical MGRS and the covering based MGRS.

Theorem 3.3. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U and $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq U$. Then,

(1) $\underline{\sum_{i=1}^m \mathcal{C}_i}^O (X_1) \subseteq \underline{\sum_{i=1}^m \mathcal{C}_i}^O (X_2) \subseteq \cdots \subseteq \underline{\sum_{i=1}^m \mathcal{C}_i}^O (X_n)$,

(2) $\overline{\sum_{i=1}^m \mathcal{C}_i}^O (X_1) \subseteq \overline{\sum_{i=1}^m \mathcal{C}_i}^O (X_2) \subseteq \cdots \subseteq \overline{\sum_{i=1}^m \mathcal{C}_i}^O (X_n)$.

Proof. They can be easily proved by Definition 3.1.

Theorem 3.4. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U and $X \subseteq U$. Suppose $\mathcal{C}_1 \preceq^c \mathcal{C}_2 \preceq^c \cdots \preceq^c \mathcal{C}_m$, then,

(1) $\underline{\sum_{i=1}^m \mathcal{C}_i}^O (X) = \underline{\mathcal{C}_m}(X)$,

(2) $\overline{\sum_{i=1}^m \mathcal{C}_i}^O (X) = \overline{\mathcal{C}_m}(X)$.

Proof. (1) For any $x \in \underline{\sum_{i=1}^m \mathcal{C}_i}^O (X)$, we have $(K_{ij})_x \subseteq X$, where $i \in \{1, 2, \cdots, m\}$ and $j = 1, 2, \cdots, t_i$. Note that $\mathcal{C}_1 \preceq^c \mathcal{C}_2 \preceq^c \cdots \preceq^c \mathcal{C}_m$. There must exist $(K_{pq})_x \in \mathcal{C}_m$ such that $(K_{pq})_x \subseteq X$, where $p \in \{1, 2, \cdots, m\}$ and $q \in \{1, 2, \cdots, t_i\}$. It follows that $x \in \underline{\mathcal{C}_m} X$. Hence, $\underline{\sum_{i=1}^m \mathcal{C}_i}^O X \subseteq \underline{\mathcal{C}_m} X$. On the other hand, for any $x \in \underline{\mathcal{C}_m} X$, we have that

$(K_{mj})_x \subseteq X$, where $j \in \{1, 2, \cdots, t_i\}$. Moreover, according to $\mathcal{C}_1 \preceq^c \mathcal{C}_2 \preceq^c \cdots \preceq^c \mathcal{C}_m$, we have $x \in (K_{1j_1})_x \subseteq (K_{2j_2})_x \subseteq \cdots \subseteq (K_{mj_m})_x \subseteq X$, where $j_l \in \{1, 2, \cdots, t_i\}$ and $l \in 1, 2, \cdots m$. By Definition 3.1, we have $x \in \underline{\sum_{i=1}^m \mathcal{C}_i}(X)$. Therefore, $\underline{\sum_{i=1}^m \mathcal{C}_i}^O X \supseteq \underline{\mathcal{C}_m} X$. Consequently, $\underline{\sum_{i=1}^m \mathcal{C}_i}^O(X) = \underline{\mathcal{C}_m} X$.

(2) Suppose that $(K_{1j_1})_x \in \mathcal{C}_1, (K_{2j_2})_x \in \mathcal{C}_2, \cdots, (K_{mj_m})_x \in \mathcal{C}_m$. By Definition 3.1, we have that $(K_{ij})_x \cap X \neq \emptyset$, where $i \in \{1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots, t_i\}$. Note that $\mathcal{C}_1 \preceq^c \mathcal{C}_2 \preceq^c \cdots \preceq^c \mathcal{C}_m$. Hence, $(K_{1j_1})_x \preceq (K_{2j_2})_x \preceq \cdots \preceq (K_{mj_m})_x$. By Theorem 3.3, $\overline{\sum_{i=1}^m \mathcal{C}_i}^O(X) = \overline{\mathcal{C}_m}(X)$.

Example 3.4 (Continued from Example 3.2). Let $X = \{x_1, x_2, x_3, x_4\}, \mathcal{C}_1 = \{\{x_1\}, \{x_2\}, \{x_3, x_5, x_6\}, \{x_6, x_7, x_8, x_9\}\}$, $\mathcal{C}_2 = \{\{x_1, x_3\}, \{x_2, x_4\}, \{x_3, x_4, x_5, x_6, x_7, x_8, x_9\}\}$ be two coverings of U, and $\mathcal{C}_1 \preceq^c \mathcal{C}_2$. We have that $\underline{\mathcal{C}_1}(X) = \{x_1, x_2\}, \overline{\mathcal{C}_1}(X) = \{x_1, x_2, x_3, x_5, x_6\}$, $\underline{\mathcal{C}_2}(X) = \{x_1, x_2, x_3, x_4\}$, and $\overline{\mathcal{C}_2}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$. Hence, $\underline{\mathcal{C}_1 + \mathcal{C}_2}^O(X) = \{x_1, x_2, x_3, x_4\} = \underline{\mathcal{C}_2}(X)$ and $\overline{\mathcal{C}_1 + \mathcal{C}_2}^O(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \overline{\mathcal{C}_2}(X)$.

3.1.2. *The first type of pessimistic CMGRS*

Definition 3.2. Let (U, Ω) be a covering approximation space, $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ a family of coverings of U with $\mathcal{C}_i = \{K_{i1}, K_{i2}, \cdots, K_{it_i}\}$, and $X \subseteq U$. Then, a pessimistic lower approximation and a pessimistic upper approximation of X with respect to Ω are denoted by $\underline{\sum_{i=1}^m \mathcal{C}_i}^P(X)$ and $\overline{\sum_{i=1}^m \mathcal{C}_i}^P(X)$, respectively, where

$$\underline{\sum_{i=1}^m \mathcal{C}_i}^P(X) = \bigcup \{K_{ij} \in \mathcal{C}_i \mid \wedge_{i=1}^m (K_{ij} \subseteq X), i \in \{1, 2, \cdots, m\}, j = 1, 2, \cdots, t_i\}, \tag{4}$$

$$\overline{\sum_{i=1}^m \mathcal{C}_i}^P(X) = \sim \underline{\sum_{i=1}^m \mathcal{C}_i}^P(\sim X). \tag{5}$$

And the *area of uncertainty* or *boundary region* of X relative to Ω in covering based multigranulation rough sets is

$$Bn^P_{\sum_{i=1}^m \mathcal{C}_i}(X) = \overline{\sum_{i=1}^m \mathcal{C}_i}^P(X) \backslash \underline{\sum_{i=1}^m \mathcal{C}_i}^P(X).$$

Then, $(\underline{\sum_{i=1}^m \mathcal{C}_i}^P(X), \overline{\sum_{i=1}^m \mathcal{C}_i}^P(X))$ is called the first type of covering based pessimistic multigranulation rough sets (or the first type of pessimistic CMGRS, for short). We say $(U, \mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m)$ a pessimistic multigranulation covering approximation space, denoted by $((U, \sum^P \mathcal{C}_i))$ i.e., PMCA-Space .

Remark 2. In particular, when $i = 1$, the first type of pessimistic CMGRS will degenerate into a single covering based rough sets whose lower and upper approximation operations are just (1) of Definition 2.3. In addition, if $\mathcal{C}_i (i \in \{1, 2, \cdots, m\})$ is a partition on the universe U, then $(\underline{\sum_{i=1}^m \mathcal{C}_i}^P(X), \overline{\sum_{i=1}^m \mathcal{C}_i}^P(X))$ will degenerate into the original MGRS. According to Yao's opinion [54], we say that this pair of approximation operators is defined by the granule.

Theorem 3.5. Let (U,Ω) be a covering approximation space, $\Omega=\{\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{C}_m\}$ a family of coverings of U, and $X\subseteq U$. Then,

$$\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)=\{x\in U|\vee((K_{ij})_x\cap X\neq\emptyset),i\in\{1,2,\cdots,m\},j=1,2,\cdots,t_i\},\qquad(6)$$

Proof.
$$\begin{aligned}x\in\overline{\textstyle\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)&\Leftrightarrow x\in\sim\underline{\textstyle\sum_{i=1}^{m}\mathcal{C}_i}^{P}(\sim X)\\&\Leftrightarrow x\notin\underline{\textstyle\sum_{i=1}^{m}\mathcal{C}_i}^{P}(\sim X)\\&\Leftrightarrow\vee((K_{ij})_x\nsubseteq\sim X),i\in\{1,2,\cdots,m\},j=1,2,\cdots,t_i\\&\Leftrightarrow\vee(K_{ij}(x)\cap X\neq\emptyset),i\in\{1,2,\cdots,m\},j=1,2,\cdots,t_i.\end{aligned}$$

Example 3.5 (Continued from Example 3.1). By Definition 3.2, we have $\underline{(\mathcal{C}_1+\mathcal{C}_2)}^{P}(X)=\{x_2,x_5,x_8\}\cap\emptyset=\emptyset$ and $\overline{(\mathcal{C}_1+\mathcal{C}_2)}^{P}(X)=\overline{\mathcal{C}_1}X\cup\overline{\mathcal{C}_2}(X)=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9\}$. By Example 3.2, we have $\underline{\mathcal{C}_1\cap\mathcal{C}_2}(X)=\{x_1,x_2,x_5,x_8\}$, $\overline{\mathcal{C}_1\cap\mathcal{C}_2}(X)=\{x_1,x_2,x_4,x_5,x_6,x_7,x_8\}$. Hence, $\underline{(\mathcal{C}_1+\mathcal{C}_2)}^{P}(X)\subseteq\underline{\mathcal{C}_1\cap\mathcal{C}_2}(X)$ and $\overline{(\mathcal{C}_1+\mathcal{C}_2)}^{P}(X)\supseteq\overline{\mathcal{C}_1\cap\mathcal{C}_2}(X)$.

As a result of this example, we see that the pessimistic lower approximation of X induced by $\mathcal{C}_1+\mathcal{C}_2$ is not bigger than that induced by $\mathcal{C}_1\cap\mathcal{C}_2$. For a more general case, we have the following propositions.

Proposition 3.3. Let (U,Ω) be a covering approximation space, $\Omega=\{\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{C}_m\}$ a family of coverings of U, and $X\subseteq U$. Then, the following properties hold

(1) $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)\subseteq\underline{\bigcap_{i=1}^{m}\mathcal{C}_i}(X)$,

(2) $\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)\supseteq\overline{\bigcap_{i=1}^{m}\mathcal{C}_i}(X)$.

Proof. (1) For any $x\in\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)$, by Definition 3.2, it follows that there must exist $(K_{1j_1})_x\in\mathcal{C}_1,(K_{2j_2})_x\in\mathcal{C}_2,\cdots,(K_{mj_m})_x\in\mathcal{C}_m$. In fact, $x\in(K_{ij})_x$ for $i\in\{1,2,\cdots,m\},j\in\{1,2,\cdots,t_i\}$. Hence, $x\in\bigcap_{i=1}^{m}K_{it_i}(x)$. Note that $\bigcap_{i=1}^{m}(K_{ij})_x\subseteq\Omega_x$ for any $x\in U$, and $\underline{\bigcap_{i=1}^{m}\mathcal{C}_i}X=\bigcup\{\Omega_x\mid\Omega_x\subseteq X\}$. As a result, $x\in\underline{\bigcap_{i=1}^{m}\mathcal{C}_i}X$.

(2) For any $x\in\overline{\bigcap_{i=1}^{m}\mathcal{C}_i}(X)$, there exists Ω_x such that $x\in\Omega_x$ and $\Omega_x\cap X\neq\emptyset$. Note that $\Omega_x\subseteq(K_{ij})_x$, where $i\in\{1,2,\cdots,m\}$ and $j\in\{1,2,\cdots,t_i\}$. Hence, $(K_{ij})_x\cap X\neq\emptyset$, i. e., $x\in\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)$. Therefore, $\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)\supseteq\overline{\bigcap_{i=1}^{m}\mathcal{C}_i}(X)$.

Proposition 3.4. Let $\Omega=\{\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{C}_m\}$ be a family of coverings of U, and $X,Y\subseteq U$. Then, the following properties hold

(1) $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}U=\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}U=U$,

(2)$\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}\emptyset=\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}\emptyset=\emptyset$,

(3) $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)\subseteq X\subseteq\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)$,

(4) $\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X\cup Y)=\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)\cup\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{p}(Y)$,

(5) $X\subseteq Y\Rightarrow\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)\subseteq\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(Y)$,

(6) $X\subseteq Y\Rightarrow\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)\subseteq\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(Y)$,

(7) $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)=\sim\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(\sim X)$,

(8) $\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)=\sim\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(\sim X)$.

Proof. These can be easily proved by Definition 3.2.

However, some propositions held in the original MGRS cannot hold in the covering based pessimistic multigranulation rough sets. For example,

(1) $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X\cap Y)=\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)\cap\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(Y)$,
(2) $\overline{\overline{\sum_{i=1}^{m}\mathcal{C}_i}}^P(X)=\overline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)$.

Now, we use two counter-examples to confirm our assertions.

(1) Continued from Example 3.3, let $X=\{x_1,x_2,x_5,x_8\}$, $Y=\{x_2,x_5,x_7,x_8,x_9\}\subseteq U$, $\mathcal{C}_1=\{\{x_1\},\{x_2\},\{x_3,x_5,x_6\},\{x_6,x_7,x_8,x_9\}\}$ and $\mathcal{C}_2=\{\{x_1,x_2,x_5\},\{x_2,x_7\},\{x_3,x_4,$ $x_5,x_6,x_7,x_8,x_9\}\}$ two coverings on U. By Definition 3.2, we can get $X\cap Y=\{x_2,x_5,x_8\}$. Then $\underline{\mathcal{C}_1+\mathcal{C}_2}^P(X\cap Y)=\emptyset$, $\underline{\mathcal{C}_1+\mathcal{C}_2}^P(X)=\{x_2,x_1\}$, $\underline{\mathcal{C}_1+\mathcal{C}_2}^P(Y)=\{x_2,x_7\}$, and $\underline{(\mathcal{C}_1+\mathcal{C}_2)}^P(X)\cap\underline{(\mathcal{C}_1+\mathcal{C}_2)}^P(Y)=\{x_2\}$. Hence, (1) does not hold, i.e., $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X\cap Y)\neq\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)\cap\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(Y)$.

(2) Continued from Example 3.3, let $X=\{x_4\}$. Now, $\overline{\mathcal{C}_1+\mathcal{C}_2}^P(X)=\{x_1,x_2,x_4,x_5,x_6,$ $x_7,x_8,x_3\}$ and $\overline{\overline{\mathcal{C}_1+\mathcal{C}_2}}^P(X)=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9\}$. Obviously, (2) also does not hold, i.e., $\overline{\overline{\sum_{i=1}^{m}\mathcal{C}_i}}^P(X)\neq\overline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)$.

Also, these counter-examples show a distinction between the classical MGRS and the covering based MGRS.

Theorem 3.6. Let $\Omega=\{\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{C}_m\}$ be a family of coverings of U and $X_1\subseteq X_2\subseteq\cdots\subseteq X_n\subseteq U$. Then,
(1) $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X_1)\subseteq\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X_2)\subseteq\cdots\subseteq\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X_n)$,
(2) $\overline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X_1)\subseteq\overline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X_2)\subseteq\cdots\subseteq\overline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X_n)$.

Proof . These can be proved by Definition 3.2.

Theorem 3.7. Let $\Omega=\{\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{C}_m\}$ be a family of coverings of U and $X\subseteq U$. If $\mathcal{C}_1\preceq^c\mathcal{C}_2\preceq^c\cdots\preceq^c\mathcal{C}_m$, then,
(1) $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)=\underline{\mathcal{C}_m}(X)$,
(2) $\overline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)=\overline{\mathcal{C}_m}(X)$.

Proof. (1) For any $x\in\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)$, we have $(K_{ij})_x\subseteq X$. For $i=1,2,\cdots,m$, it follows $x\in\underline{\mathcal{C}_m}(X)$. For any $x\in\underline{\mathcal{C}_m}(X)$, we have $(K_{mj})_x\subseteq X$. By $\mathcal{C}_1\preceq^c\mathcal{C}_2\preceq^c\cdots\preceq^c\mathcal{C}_m$, we have that $(K_{1j_1})_x\subseteq(K_{2j_2})_x\subseteq\cdots\subseteq(K_{mj_m})_x\subseteq X$. According to Definition 3.4, we obtain $x\in\underline{\sum_{i=1}^{m}\mathcal{C}_i}(X)$. Hence, $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)=\underline{\mathcal{C}_m}(X)$. Similarly, (2) can be proved.

Finally, it is necessary to discuss the relationship between the above two different covering based multigranulation rough sets.

Theorem 3.8. Let $\Omega=\{\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{C}_m\}$ be a family of coverings of U and $X\subseteq U$. The optimistic and pessimistic covering based multigranulation rough sets are denoted by $(\underline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X),\overline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X))$ and $(\underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X),\overline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X))$, respectively. Then, the following properties hold

(1) $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X) \supseteq \underline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)$,

(2) $\overline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X) \subseteq \overline{\sum_{i=1}^{m}\mathcal{C}_i}^P(X)$.

Proof. They can be proved by Theorem 3.1 and Theorem 3.5.

Example 3.6 (Continued from Example 3.2). From Example 3.2, we have six coverings of U. Let $X = \{x_1, x_2, x_5, x_8, x_9\} \subseteq U$. By Definition 3.1, we have that $\underline{\sum_{i=1}^{6}\mathcal{C}_i}^O(X) = \{x_2, x_5, x_8, x_9\}$ and $\underline{\sum_{i=1}^{6}\mathcal{C}_i}^P(X) = \{\{x_2, x_8\} \cap \{x_2, x_5, x_9\}\} = \{x_2\}$. Hence, $\underline{\sum_{i=1}^{6}\mathcal{C}_i}^O(X) \supseteq \underline{\sum_{i=1}^{6}\mathcal{C}_i}^P X$. Similarly, $\overline{\sum_{i=1}^{6}\mathcal{C}_i}^O(X) = \{x_1, x_2, x_3, x_5, x_6, x_8, x_9\}$ and $\overline{\sum_{i=1}^{6}\mathcal{C}_i}^P(X) = U$. Hence, $\overline{\sum_{i=1}^{6}\mathcal{C}_i}^O(X) \subseteq \overline{\sum_{i=1}^{6}\mathcal{C}_i}^P(X)$.

3.2. *The second type of covering approximation operators based multigranulation rough sets (Or the second type of CMGRS)*

3.2.1. *The second type of optimistic CMGRS*

Definition 3.3. Let (U, Ω) be a covering approximation space, $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ a family of coverings of U with $\mathcal{C}_i = \{N_{i1}(x_1), N_{i2}(x_2), \cdots, N_{it_i}(x_{|U|})\}$, and $X \subseteq U$. An optimistic lower approximation and an optimistic upper approximation of X with respect to Ω, denoted by $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X)$ and $\overline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X)$, are defined as

$$\underline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X) = \{x \in U \mid \vee(N_{ij}(x) \subseteq X), i \in \{1, 2, \cdots, m\}, j = 1, 2, \cdots, t_i\}, \tag{7}$$

$$\overline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X) = \{x \in U \mid \wedge(N_{ij}(x) \cap X \neq \emptyset), i \in \{1, 2, \cdots, m\}, j = 1, 2, \cdots, t_i\}, \tag{8}$$

where $N(x) = \cap\{K \in \mathcal{C} \mid x \in K\}$.

And the *area of uncertainty* or *boundary region* of X relative to Ω in covering based multigranulation rough sets is

$$Bn^O_{\sum_{i=1}^{m}\mathcal{C}_i}(X) = \overline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X) \backslash \underline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X).$$

Then, $(\underline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X), \overline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X))$ is called the second type of covering based optimistic multigranulation rough sets (or the second type of optimistic CMGRS, for short).

In particular, when $i = 1$, the second type of optimistic CMGRS will degenerate into the second type of covering approximation operators listed in Section 2. Additionally, if $\mathcal{C}_i, i \in \{1, 2, \cdots, m\}$ is a partition on the universe U, then $(\underline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X), \overline{\sum_{i=1}^{m}\mathcal{C}_i}^O(X))$ will degenerate into the original MGRS. According to Yao's opinion [54], we say that this pair of approximation operators is defined by the element based definition.

3.2.2. *The second type of pessimistic CMGRS*

Definition 3.4. Let (U,Ω) be a covering approximation space, $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ a family of coverings of U with $\mathcal{C}_i = \{N_{i1}(x_1), N_{i2}(x_2), \cdots, N_{it_i}(x_{|U|})\}$, and $X \subseteq U$. Then, a pessimistic lower approximation and a pessimistic upper approximation of X with respect to Ω are denoted by $\underline{\sum_{i=1}^m \mathcal{C}_i}^P(X)$ and $\overline{\sum_{i=1}^m \mathcal{C}_i}^P(X)$, respectively, where

$$\underline{\sum_{i=1}^{m} \mathcal{C}_i}^P(X) = \{x \in U \mid \wedge_{i=1}^m (N_{ij}(x) \subseteq X), i \in \{1,2,\cdots,m\}, j = 1,2,\cdots,t_i\}, \quad (9)$$

$$\overline{\sum_{i=1}^{m} \mathcal{C}_i}^P(X) = \{x \in U | \vee (N_{ij}(x) \cap X \neq \emptyset)\}. \quad (10)$$

And the *area of uncertainty* or *boundary region* of X relative to Ω in covering based multigranulation rough sets is

$$Bn^P_{\sum_{i=1}^m \mathcal{C}_i}(X) = \overline{\sum_{i=1}^{m} \mathcal{C}_i}^P(X) \backslash \underline{\sum_{i=1}^{m} \mathcal{C}_i}^P(X).$$

Then, $(\underline{\sum_{i=1}^m \mathcal{C}_i}^P(X), \overline{\sum_{i=1}^m \mathcal{C}_i}^P(X))$ is called the second type of covering based pessimistic multigranulation rough sets (or the second type of pessimistic CMGRS, for short).

In a special case, when $i = 1$, the second type of pessimistic CMGRS will degenerate into the second type of covering approximation operators listed in this paper. Additionally, if $\mathcal{C}_i, i \in \{1,2,\cdots,m\}$ is a partition on the universe U, then $(\underline{\sum_{i=1}^m \mathcal{C}_i}^O(X), \overline{\sum_{i=1}^m \mathcal{C}_i}^O(X))$ will degenerate into the original MGRS. Here, the properties of the second type of optimistic and pessimistic CMGRS are omitted.

3.3. *The third type of covering approximation operators based multigranulation rough sets (Or the third type of CMGRS)*

3.3.1. *The third type of optimistic CMGRS*

Definition 3.5. Let (U,Ω) be a covering approximation space, $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ a family of coverings of U with $\mathcal{C}_i = \{K_{i1}, K_{i2}, \cdots, K_{it_i}\}$, and $X \subseteq U$. An optimistic lower approximation and an optimistic upper approximation of X with respect to Ω, denoted by $\underline{\sum_{i=1}^m \mathcal{C}_i}^O(X)$ and $\overline{\sum_{i=1}^m \mathcal{C}_i}^O(X)$, are defined as

$$\underline{\sum_{i=1}^{m} \mathcal{C}_i}^O(X) = \bigcup\{K_{ij} \in \mathcal{C}_i \mid \vee(K_{ij} \subseteq X), i \in \{1,2,\cdots,m\}, j = 1,2,\cdots,t_i\}, \quad (11)$$

$$\overline{\sum_{i=1}^{m} \mathcal{C}_i}^O(X) = \{x \in U \mid \wedge((K_{ij})_x \cap X \neq \emptyset), i \in \{1,2,\cdots,m\}, j = 1,2,\cdots,t_i\}, \quad (12)$$

where $t_i = |\mathcal{C}_i|$.

And the *area of uncertainty* or *boundary region* of X relative to Ω in covering based multigranulation rough sets is

$$Bn^{O}_{\sum_{i=1}^{m}\mathcal{C}_i}(X) = \overline{\sum_{i=1}^{m}\mathcal{C}_i}^{O}(X) \backslash \underline{\sum_{i=1}^{m}\mathcal{C}_i}^{O}(X).$$

Then, $(\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{O}(X), \overline{\sum_{i=1}^{m}\mathcal{C}_i}^{O}(X))$ is called the third type of covering based optimistic multigranulation rough sets (or the third type of optimistic CMGRS, for short). In a special case, when $i = 1$, the third type of optimistic CMGRS will degenerate into the third type of covering approximation operators listed in this paper. Additionally, if $\mathcal{C}_i, i \in \{1, 2, \cdots, m\}$ is a partition on the universe U, then $(\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{O}(X), \overline{\sum_{i=1}^{m}\mathcal{C}_i}^{O}(X))$ will degenerate into the original MGRS.

3.3.2. *The third type of pessimistic CMGRS*

Definition 3.6. Let (U, Ω) be a covering approximation space, $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ a family of coverings of U with $\mathcal{C}_i = \{K_{i1}, K_{i2}, \cdots, K_{it_i}\}$, and $X \subseteq U$. Then, a pessimistic lower approximation and a pessimistic upper approximation of X with respect to Ω are denoted by $\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)$ and $\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X)$, respectively, where

$$\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X) = \bigcup\{K_{ij} \in \mathcal{C}_i \mid \wedge_{i=1}^{m}(K_{ij} \subseteq X), i \in \{1, 2, \cdots, m\}, j = 1, 2, \cdots, t_i, x \in U\}, \tag{13}$$

$$\overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X) = \{x \in U \mid \vee((K_{ij})_x \cap X \neq \emptyset), i \in \{1, 2, \cdots, m\}, j = 1, 2, \cdots, t_i, x \in U\}. \tag{14}$$

And the *area of uncertainty* or *boundary region* of X relative to Ω in covering based multigranulation rough sets is

$$Bn^{P}_{\sum_{i=1}^{m}\mathcal{C}_i}(X) = \overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X) \backslash \underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X).$$

Then, $(\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X), \overline{\sum_{i=1}^{m}\mathcal{C}_i}^{P}(X))$ is called as the third type of covering based pessimistic multigranulation rough sets (or the third type of pessimistic CMGRS, for short).

In a special case, when $i = 1$, the third type of pessimistic CMGRS will degenerate into the third type of covering approximation operators listed in Section 2. Additionally, if $\mathcal{C}_i(i \in \{1, 2, \cdots, m\})$ is a partition of the universe U, then $(\underline{\sum_{i=1}^{m}\mathcal{C}_i}^{O}(X), \overline{\sum_{i=1}^{m}\mathcal{C}_i}^{O}(X))$ will degenerate into the original MGRS. Here, the properties of the third type of optimistic and pessimistic CMGRSs are omitted.

Corresponding to the properties of MGRS listed in Section 2.2, the proposed covering based optimistic multigranulation rough sets can be summarized in Table 2.

Table 2
The properties of three types of covering based optimistic multigranulation rough sets

MCA-Space $(U, \sum^O \mathcal{C}_i)$	Satisfied	Not satisfied
$(U, \sum^O \mathcal{C}_1)$	(1OML),(2OML),(3OML),(4OML),(5OML) (6OML),(7OML),(8OML),(9OML),(10OML) (1OMH),(2OMH),(3OMH),(4OMH),(5OMH) (7OMH),(8OMH),(9OMH),(10OMH)	(6OMH)
$(U, \sum^O \mathcal{C}_2)$	(1OML),(2OML),(3OML),(4OML),(5OML) (6OML),(7OML),(8OML),(9OML),(10OML) (1OMH),(2OMH),(3OMH),(4OMH),(5OMH) (7OMH),(8OMH),(9OMH),(10OMH)	(6OMH)
$(U, \sum^O \mathcal{C}_3)$	(1OML),(2OML),(3OML),(4OML),(5OML) (6OML),(8OML),(9OML),(10OML) (1OMH),(2OMH),(3OMH),(4OMH) (5OMH),(8OMH),(9OMH),(10OMH)	(7OML) (6OMH) (7OMH)

In Table 2, $(U, \sum^O \mathcal{C}_i)$ represents the i-th($i \in \{1, 2, 3\}$) type of covering based optimistic multigranulation approximation space. Similarly, the proposed covering based pessimistic multigranulation rough sets can also be summarized in Table 3.

Table 3
The properties of three types of covering based pessimistic multigranulation rough sets

MCA-Space $(U, \sum^P \mathcal{C}_i)$	Satisfied	Not satisfied
$(U, \sum^P \mathcal{C}_1)$	(1PML),(2PML),(3PML),(5PML) (7PML),(8PML),(9PML),(10PML) (1PMH),(2PMH),(3PMH),(5PMH) (7PMH),(8PMH),(9PMH),(10PMH)	(4PML) (4PMH) (6PMH)
$(U, \sum^P \mathcal{C}_2)$	(1PML),(2PML),(3PML),(5PML) (7PML),(8PML),(9PML),(10PML) (1PMH),(2PMH),(3PMH),(5PMH) (7PMH),(8PMH),(9PMH),(10PMH)	(4PML) (4PMH) (6PMH)
$(U, \sum^P \mathcal{C}_3)$	(1PML),(2PML),(3PML), (5PML) (8PML),(9PML),(10PML),(1PMH) (2PMH),(3PMH),(5PMH) (8PMH),(10PMH)	(4PML) (7PML) (4PMH),(9PMH) (6PMH),(7PMH)

In Table 3, $(U, \sum^P \mathcal{C}_i)$ represents the i-th($i \in \{1, 2, 3\}$) type of covering based pessimistic multigranulation approximation space.

Remark 3. In this section, we have proposed three types of covering based optimistic and pessimistic multigranulation rough sets and discussed some relationships between the CMGRS and the original MGRS. Several results held in the original MGRS model but cannot hold in all the three CMGRSs. It can be known from the above discussions that (1) the original MGRS is a special case of the CMGRS and the latter degenerates into the former when each covering is a partition on the universe, and (2) compared with the original MGRS, the CMGRS theory has its advantage in the application scope since it is applicable to the covering environment, which is beneficial to the application of the idea of multigranulation for knowledge representation, rule acquisition and feature selection from a multi-source covering information system.

4. Uncertainty measures of covering based multigranulation rough sets

Multigranulation rough set (MGRS) theory is a relatively mathematical tool for solving complex problems in the multiple granulations or distributed circumstances through determining their vagueness and uncertainty. However, the existing uncertainty measures of a single covering granulation based rough sets [9, 18, 39-41, 49] are no longer suitable for covering based multigranulation rough sets. In this section, we will introduce some measures to characterize the vagueness and uncertainty of these new rough set models. Thus, these new rough set theories will contribute a lot to the applications in the fields of pattern recognition, image processing, and fuzzy reasoning.

We notice that in the literature [30], Pawlak has given a definition of the rough membership as follows.

Definition 4.1. Let $S = (U, AT)$ be an information system. For $A \subseteq AT$, $X \subseteq U$, the rough membership of x in X is defined by

$$\mu_X^A(x) = \frac{|[x]_A \cap X|}{|[x]_A|},$$

where $[x]_A$ represents an equivalence class induced by an attribute set A.

However, it is not be suitable to evaluate the uncertainty of a covering based rough sets. So the new definition of rough membership of x in X is needed.

Definition 4.2. Let $S = (U, AT)$ be an information system, $\mathcal{C}$ a covering of the universe U where $\mathcal{C} = \{K_1, K_2, \cdots, K_t\}$ and $X \subseteq U$. The maximal and minimal covering based rough memberships of x in K, denoted by $\mu_X^{\mathcal{C}}(x)$, $\eta_X^{\mathcal{C}}(x)$, are defined by

$$\mu_X^{\mathcal{C}}(x) = max\frac{|(K_i)_x \cap X|}{|K_{i_x}|},$$

$$\eta_X^{\mathcal{C}}(x) = min\frac{|(K_i)_x \cap X|}{|(K_i)_x|},$$

where $(K_i)_x \in \mathcal{C}$ and $x \in (K_i)_x$.

Proposition 4.1. Let $S = (U, AT)$ be an information system, $\mathcal{C}$ a covering of the universe U where $\mathcal{C} = \{K_1, K_2, \cdots, K_t\}$ and $X \subseteq U$. Then, the following properties hold

(1) $\mu_X^{\mathcal{C}}(x) = 1 \Leftrightarrow \exists (K_i)_x \in \mathcal{C} \wedge (K_i)_x \subseteq X$,

(2) $0 < \mu_X^{\mathcal{C}}(x) \leq 1 \Leftrightarrow \exists (K_i)_x \in \mathcal{C} \wedge (K_i)_x \cap X \neq \phi$,

(3) $\eta_X^{\mathcal{C}}(x) = 1 \Leftrightarrow \forall (K_i)_x \in \mathcal{C} \wedge (K_i)_x \subseteq X$,

(4) $0 < \eta_X^{\mathcal{C}}(x) \leq 1 \Leftrightarrow \exists (K_i)_x \in \mathcal{C} \wedge (K_i)_x \subseteq X$.

Proof. They can be easily proved by Definition 4.2.

Example 4.1 (Continued from Example 3.2). Let $\mathcal{C}_1 = \{\{x_1, x_2, x_4, x_5\}, \{x_2, x_5\}, \{x_3, x_5\}\}$ be a covering of U, and $X = \{x_1, x_3, x_5\} \subseteq U$. By Definition 4.2, we have that $\mu_X^{\mathcal{C}_1}(x_1) = max\{\frac{|\{x_1,x_2,x_4,x_5\}\cap X|}{|\{x_1,x_2,x_4,x_5\}|}\} = \frac{1}{2}$, $\mu_X^{\mathcal{C}_1}(x_2) = max\{\frac{|\{x_1,x_2,x_4,x_5\}\cap X|}{|\{x_1,x_2,x_4,x_5\}|}, \frac{|\{x_2,x_5\}\cap X|}{|\{x_1,x_2,x_4,x_5\}|}\} = max\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$, $\mu_X^{\mathcal{C}_1}(x_3) = 1$, $\mu_X^{\mathcal{C}_1}(x_4) = \frac{1}{2}$, and $\mu_X^{\mathcal{C}_1}(x_5) = max\{\frac{1}{2}, \frac{1}{2}, 1\} = 1$. Similarly, we have that $\eta_X^{\mathcal{C}_1}(x_1) = \frac{1}{2}$, $\eta_X^{\mathcal{C}_1}(x_2) = \frac{1}{2}$, $\eta_X^{\mathcal{C}_1}(x_3) = 1$, $\eta_X^{\mathcal{C}_1}(x_4) = \frac{1}{2}$, and $\eta_X^{\mathcal{C}_1}(x_5) = min\{\frac{1}{2}, \frac{1}{2}, 1\} = \frac{1}{2}$.

Definition 4.3. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U and $X \subseteq U$. The maximal and minimal degree of rough membership of x in X, denoted by $\mu_X^{\Omega}(x)$ and $\eta_X^{\Omega}(x)$ are defined by

$$\mu_X^{\Omega}(x) = \frac{1}{m}\sum_{i=1}^{m} \mu_X^{\mathcal{C}_i}(x),$$

$$\eta_X^{\Omega}(x) = \frac{1}{m}\sum_{i=1}^{m} \eta_X^{\mathcal{C}_i}(x).$$

Proposition 4.2. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U and $X \subseteq U$. Then, we have that

(1) $0 < \mu_X^{\Omega}(x) \leq 1$,

(2) $0 < \eta_X^{\Omega}(x) \leq 1$.

Proof. They can be proved by Definition 4.3.

Example 4.2 (Continued from Example 3.1). Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2\}$ be a family of coverings of U, where $\mathcal{C}_1 = \{\{x_1, x_2, x_4, x_5\}, \{x_2, x_5\}, \{x_3, x_5\}\}$ and $\mathcal{C}_2 = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_2, x_4\}\}$. For $X = \{x_1, x_3, x_5\} \subseteq U$, by Definition 4.2, we have that $\mu_X^{\mathcal{C}_2}(x_1) = \frac{2}{3}$, $\mu_X^{\mathcal{C}_2}(x_2) = \frac{2}{3}$, $\mu_X^{\mathcal{C}_2}(x_3) = \frac{2}{3}$, $\mu_X^{\mathcal{C}_2}(x_4) = \frac{1}{2}$, and $\mu_X^{\mathcal{C}_2}(x_5) = 0$. Similarly, we have that $\eta_X^{\mathcal{C}_2}(x_1) = \frac{2}{3}$, $\eta_X^{\mathcal{C}_2}(x_2) = 0$, $\eta_X^{\mathcal{C}_2}(x_3) = \frac{2}{3}$, $\eta_X^{\mathcal{C}_2}(x_4) = 0$, and $\eta_X^{\mathcal{C}_2}(x_5) = 0$. According to the results obtained from Example 4.1 and Definition 4.3, we get that

$\mu_X^{\Omega}(x_1) = \frac{\mu_X^{\mathcal{C}_1}(x_1)+\mu_X^{\mathcal{C}_2}(x_1)}{2} = \frac{1}{2} \times (\frac{1}{2} + \frac{2}{3}) = \frac{7}{12}$,
$\mu_X^{\Omega}(x_2) = \frac{\mu_X^{\mathcal{C}_1}(x_2)+\mu_X^{\mathcal{C}_2}(x_2)}{2} = \frac{1}{2} \times (\frac{1}{2} + \frac{2}{3}) = \frac{7}{12}$,
$\mu_X^{\Omega}(x_3) = \frac{\mu_X^{\mathcal{C}_1}(x_3)+\mu_X^{\mathcal{C}_2}(x_3)}{2} = \frac{1}{2} \times (1 + \frac{2}{3}) = \frac{5}{6}$,
$\mu_X^{\Omega}(x_4) = \frac{\mu_X^{\mathcal{C}_1}(x_4)+\mu_X^{\mathcal{C}_2}(x_4)}{2} = \frac{1}{2} \times (\frac{1}{2} + \frac{1}{2}) = \frac{1}{2}$,
$\mu_X^{\Omega}(x_5) = \frac{\mu_X^{\mathcal{C}_1}(x_5)+\mu_X^{\mathcal{C}_2}(x_5)}{2} = \frac{1}{2} \times (\frac{1}{2} + 0) = \frac{1}{4}$.
Similarly,
$\eta_X^{\Omega}(x_1) = \frac{\eta_X^{\mathcal{C}_1}(x_1)+\eta_X^{\mathcal{C}_2}(x_1)}{2} = \frac{1}{2} \times (\frac{1}{2} + \frac{2}{3}) = \frac{7}{12}$,

$\eta_X^{\Omega}(x_2) = \frac{\eta_X^{\mathcal{C}_1}(x_2)+\eta_X^{\mathcal{C}_2}(x_2)}{2} = \frac{1}{2} \times (\frac{1}{2}+0) = \frac{1}{4}$,
$\eta_X^{\Omega}(x_3) = \frac{\eta_X^{\mathcal{C}_1}(x_3)+\eta_X^{\mathcal{C}_2}(x_3)}{2} = \frac{1}{2} \times (1+\frac{2}{3}) = \frac{5}{6}$,
$\eta_X^{\Omega}(x_4) = \frac{\eta_X^{\mathcal{C}_1}(x_4)+\eta_X^{\mathcal{C}_2}(x_4)}{2} = \frac{1}{2} \times (\frac{1}{2}+0) = \frac{1}{4}$,
$\eta_X^{\Omega}(x_5) = \frac{\eta_X^{\mathcal{C}_1}(x_5)+\eta_X^{\mathcal{C}_2}(x_5)}{2} = \frac{1}{2} \times (\frac{1}{2}+0) = \frac{1}{4}$.

Definition 4.4. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U. An optimistic approximation measure of X by Ω is defined as

$$\alpha_{\Omega}^{O}(X) = \frac{|\underline{\sum_{i=1}^{m} \mathcal{C}_i}^{O} X|}{|\overline{\sum_{i=1}^{m} \mathcal{C}_i}^{O} X|},$$

where $X \neq \emptyset$ and $|X|$ denotes the cardinality of a set X. Similarly, a pessimistic approximation measure of X by Ω is defined as

$$\alpha_{\Omega}^{P}(X) = \frac{|\underline{\sum_{i=1}^{m} \mathcal{C}_i}^{P} X|}{|\overline{\sum_{i=1}^{m} \mathcal{C}_i}^{P} X|}.$$

Theorem 4.1. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U and $\Omega^{'} \subseteq \Omega$. Then $\alpha_{\Omega}^{O}(X) \geq \alpha_{\Omega^{'}}^{O}(X) \geq \alpha_{\mathcal{C}_i}(X)$ and $\alpha_{\Omega}^{P}(X) \leq \alpha_{\Omega^{'}}^{P}(X) \leq \alpha_{\mathcal{C}_i}(X)$, $(i \leq m)$.

Proof. They can be proved by Definition 4.4.

In the optimistic covering based multigranulation rough sets, the approximation measure of X by Ω is not smaller than that induced by a subset of Ω. The approximation measure of X by $\Omega^{'}$ is also not smaller than that induced by a single covering granulation. Whereas, in the pessimistic version, the result is just the converse.

Definition 4.5. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U and $\mathcal{C}_i = \{K_{i1}, K_{i2}, \cdots, K_{it_i}\}$. The rough entropy of Ω is defined by the following

$$E(\Omega) = \frac{1}{m} \sum_{i=1}^{m} E(\mathcal{C}_i),$$

where $E(\mathcal{C}_i) = \sum_{j=1}^{t_i} \frac{|K_{ij}|}{t_i} \log_2 |K_{ij}|, i \in \{1, 2, \cdots, m\}$ (see [18]).

Example 4.3 (Continued from Example 3.1). Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2\}$ be a family of coverings of U, where $\mathcal{C}_1 = \{\{x_1, x_2, x_4, x_5\}, \{x_2, x_5\}, \{x_3, x_5\}\}, \mathcal{C}_2 = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_2, x_4\}\}$. By Definition 4.5, we have that $E(\mathcal{C}_1) = \sum_{j=1}^{3} \frac{|K_{1j}|}{3} \log_2 |K_{1j}| = \frac{1}{3}(4log_2 4 + 2log_2 2 + 2log_2 2) = \frac{8}{3}$ and $E(\mathcal{C}_2) = \sum_{j=1}^{3} \frac{|K_{2j}|}{3} \log_2 |K_{2j}| = \frac{1}{3}(3log_2 3 + 2log_2 3 + 2log_2 2) = \frac{2}{3} + \frac{5}{3}log_2 3$. Hence, $E(\Omega) = \frac{5}{3} + \frac{5}{6}log_2 3$.

Theorem 4.2. Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m\}$ be a family of coverings of U. If $\mathcal{C}_1 \preceq^c \mathcal{C}_2 \preceq^c \cdots \preceq^c \mathcal{C}_m$, then $E(\mathcal{C}_1) \leq E(\Omega) \leq E(\mathcal{C}_m)$.

Proof. It can be proved by Definition 4.5.

Example 4.4 (Continued from Example 2.1). Let $\Omega = \{\mathcal{C}_1, \mathcal{C}_2\}$ be a family of coverings of U, where $\mathcal{C}_1 = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_4, x_5\}\}$, $\mathcal{C}_2 = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_2, x_3, x_4, x_5\}\}$ and $\mathcal{C}_1 \preceq^c \mathcal{C}_2$. By Definition 4.5, we have that $E(\mathcal{C}_1) = \sum_{j=1}^{4} \frac{|K_{1j}|}{4} \log_2 |K_{1j}| = \frac{1}{4}(1log_2 1+$

$1log_2 1 + 2log_2 2 + 2log_2 2) = \frac{1}{2}$ and $E(\mathcal{C}_2) = \sum_{j=1}^{3} \frac{|K_{2j}|}{3} \log_2 |K_{2j}| = \frac{1}{3}(2log_2 2 + 2log_2 2 + 2log_2 4) = \frac{4}{3}$. Hence, $E(\Omega) = \frac{1}{2}(\frac{1}{2} + \frac{4}{3}) = \frac{11}{12}$. Therefore, $E(\mathcal{C}_1) \leq E(\Omega) \leq E(\mathcal{C}_2)$.

Remark 4. In this section, we have systematically investigated the united measurement formations. Some examples have been employed to illustrate the application of these measures by the first type of CMGRS. Similarly, the proposed uncertainty measures can offer a method to characterize some other types of approximate abilities of the covering based multigranulation models, such as approximate precision, the rough membership, and the maximal and minimal degree of rough membership. Under the framework of covering based multiple granulations, these uncertainty measures may become a theoretical basis of granule reduction, granulation space reduction, and rule evaluation for a target information system with the covering background.

5. Conclusion and Discussion

The main contribution of this paper is that three types of optimistic and pessimistic covering based multigranulation rough sets have been proposed which can be used to do data analysis characterized by the covering environment. Under the framework of covering based multigranulation rough sets, we have investigated some of their important properties and compared their properties with those of the classical MGRS. In addition, we have introduced several important uncertainty measures, such as degree of rough membership, approximation measure, and rough entropy. These results can enrich the MGRS theory and enlarge its application scope to some extent.

Further research includes how to reduce redundant granules and how to reduce redundant granular space in the process of rough data analysis under the multigranulation environment. Another important issue in the future is to investigate applications of this new rough set theory for knowledge representation, rule acquisition, feature selection in knowledge discovery.

Acknowledgements

The authors would like to thank the anonymous reviewers and the editor for their constructive and valuable comments. This work is supported by grants from National Natural Science Foundation of China under Grant (Nos. 71031006, 71140004, 11061004), National Key Basic Research and Development Program of China (973) (No. 2013CB329404), Doctoral Program of Higher Education (No. 20121401110013), Innovative Talents of Higher Learning Institutions of Shanxi, China (No. 20120301), and Foundation of Fujian Province in China (No. 2012D141).

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