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Interval-valued hesitant fuzzy multi-granularity three-way decisions in consensus processes with applications to multi-attribute group decision making

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ABSTRACT

Multi-attribute group decision making (MAGDM) is a common activity for multi-variable complicated decision making situations by integrating collective wisdom. Aiming at fusing granular computing with three-way decisions (3WD) to study scheme synthesis and analysis of solution space, multi-granularity three-way decisions (MG-3WD) provide multi-dimension problem solving methods for MAGDM problems. By using MG-3WD frameworks, this paper intends to study viable strategies of processing consensus and conflicting opinions provided by different decision makers in the interval-valued hesitant fuzzy (IVHF) MAGDM problem. More specifically, after reviewing the relevant literature, four kinds of IVHF multigranulation decision-theoretic rough sets (MG-DTRSs) over two universes are proposed according to different risk appetites of experts firstly. Then, we explore some fundamental propositions of newly proposed models. Afterwards, solutions to MAGDM problems in the context of mergers and acquisitions (M&A) target selections by using the presented IVHF MG-DTRSs over two universes are constructed. At last, a M&A target selection case study, along with a sensitivity analysis and a comparative analysis, is applied to illustrate the established decision making approaches.

1. Introduction

1.1. The objectives of this paper

To date, many scholars have put forward a series of approaches to cope with MAGDM problems [30]. In general, the challenge of consensus processes must be handled reasonably at the beginning of MAGDM problems. It is worth pointing out decision makers' opinions may differ substantially, consensus processes are recognized to help decision makers reach a consensus, which are conductive to the processing of MAGDM problems.

In consensus processes, how to cope with consensus and conflicting opinions provided by different decision makers is significant in typical MAGDM problems. In specific, there often exist different common situations due to diverse risk

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appetites of decision makers. In this paper, we divide them as four typical classes: (i) all decision makers hold extreme opinions (completely risk-seeking or completely risk-averse) for group decision making situations; (ii) decision makers hold conflicting opinions for group decision making situations, and they agree to adopt groups' primary opinions; (iii) decision makers hold conflicting opinions for group decision making situations, and they agree to adopt personal primary opinions; (iv) decision makers hold conflicting opinions for group decision making situations, and they agree to adopt both groups' primary opinions and personal primary opinions. In addition, in lots of real-world decision procedures, decision makers may hesitate about their opinions, and they are also likely to use interval numbers due to lack of sufficient specific expert knowledge. Thus, we focus on the concept of IVHF sets (IVHFSs) for dealing with incomplete, imprecise and hesitant expression of attributes, and hope to tackle the open problem of processing the above-stated consensus and conflicting opinions in the IVHF context. In order to do this, we must tackle the following two challenges:

(1) How to design a MAGDM procedure that aids decision makers reach a consensus in the IVHF context, especially the information fusion scheme.
(2) How to address risk appetites of different decision makers in the developed MAGDM approaches.

For the sake of fully meeting all of the above-mentioned challenges, in consideration of unique features and merits owned by MG-3WD frameworks [21,22], focal interests of the work lie in merging several optimal soft computing tools together under the guidance of MG-3WD, and further handling the above-listed challenges toward high-quality decision results. Prior to stating our research motivations and contributions, we first revisit several core notions below.

1.2. A brief review on the development of MAGDM

MAGDM, an interdisciplinary research field which integrates group decision making (GDM) with multi-attribute decision making (MADM), generally offers schemes of obtaining group preference information by means of individual preference information and explicitly assessing diverse alternatives by means of various theoretical decision making models. There are many classical approaches that can be applied to addressing MAGDM problems, such as aggregation operator-based methods [14], TOPSIS-based methods [13], prospect theory-based methods [45], PROMETHEE-based methods [3], granular computing-based methods [38,46–48], etc. Those classical MAGDM methods are successfully adopted in plenty of real-world areas and they have strongly promoted the rapid development of social economy [9].

1.3. A brief review on the development of MG-3WD

MG-3WD, originated from the notion of 3WD initiated by Yao [32–34], provides comprehensive and effective rules for information granularity and multi-view hierarchical granularity analysis with the aid of 3WD. As a novel cognitive computing model, the objective of 3WD is to separate a universal set into three different independent areas by endowing the semantics of acceptance, noncommitment and rejection [35]. Moreover, Yao [36] broadly depicted and expounded 3WD as a trisecting-acting-outcome (TAO) model, which has important implications for the integration of human thinking with granular computing. Under the umbrella of the TAO model, three consecutive tasks are scheduled, i.e., separating a whole into three parts (trisecting), managing the three parts by different tactics (acting), and assessing the validity of previous results (outcome). Inspired by the concept of multi-granularity computing [19,20], many scholars put emphasis on exploring 3WD from multiple levels and further developing the framework of MG-3WD. In general, MG-3WD is extensively applied in numerous practical fields such as uncertainty measures and attribute reductions [7], cognitive concept learning [8], classifications [2], decision making [12,17,26,28,41,43], and so forth.

After a broad literature review, we have not found any MAGDM approach by virtue of MG-3WD can perfectly meet all of the above-listed primary challenges. Thus, we plan to integrate IVHFSs [4,6], multigranulation rough sets (MGRSs) over two universes [23] and DTRSs [5,11,16,29,31,37] together from the perspective of information representation, information fusion and information analysis respectively. In particular, the motivations of the integration procedure are summed up in the following reviews.

1.3.1. A brief review on the development of IVHFSs

For the sake of expressing incomplete, imprecise and hesitant information synchronously extracted from MAGDM problems, the notion of IVHFSs is employed as overall decision making contexts within this study. In IVHFSs, the corresponding form of membership degrees is represented as a set that contains several different finite values within $D[0, 1]$, where $D[0, 1]$ denotes the set of all closed subintervals on $[0, 1]$. Compared with other types of generalized fuzzy sets, IVHFSs excel in coping with more incomplete and imprecise information and allowing irresolute experts to participate in decision making processes.

1.3.2. A brief review on the development of MGRSs over two universes

The superiorities of MGRSs over two universes for solving MAGDM problems are manifested in two aspects. On the one hand, MGRSs over two universes provide experts with two different information fusion schemes (i.e., optimistic scheme and pessimistic scheme) in light of risk-pursuing and risk-fleeing mechanisms. On the other hand, the two-universe framework

[40,44] outweighs single-universe counterparts in depicting decision matrices according to the isomorphism between decision matrices and information systems. It is noted that rough set approaches own merits in mining some hidden knowledge in insufficient and incomplete information systems, the two-universe framework enables rough set approaches to provide a reasonable scheme on how to explain decisions with respect to scenarios under which decisions have been made. Hence, MGRSs over two universes can easily represent some inherent correlations between two kinds of objects in realistic MAGDM problems [15,24,25,27,39,42].

1.3.3. A brief review on the development of DTRSs

In light of risk appetites, the model of DTRSs is generally acknowledged as a primary way to realize fundamental ideas of 3WD. By integrating Bayesian decision procedures with rough sets, the formulation of losses triggered by selecting different actions is introduced in rough sets. Afterwards, by utilizing the introduced losses, the estimated costs of selecting corresponding actions can be obtained, and decision makers can construct rough approximations by determining the minimum-risk decision rules. Thus, the model of DTRSs is equipped with a scientific strategy to interpret risk appetites of decision makers. During the past decades, scholars have explored DTRSs in diverse scenarios including interval-valued fuzzy (IVF) situations [10,18,49,50], which lay a solid foundation for the study of corresponding MG-3WD models in the context of IVHFSs.

1.4. The motivations of this paper

In accordance with the above-stated motivations of the integration procedure, despite the concepts of IVHFSs, MGRSs over two universes and DTRSs play a great part in information representation, information fusion and information analysis respectively, the lack of integrative and interaction studies on these three models precludes a viable mathematical formulation of MG-3WD for handling the challenges stated in Section 1.1.

In this work, we plan to study a comprehensive and hybrid model titled IVHF MG-DTRSs over two universes based on different risk appetites of experts in a group, and eventually establish convincing IVHF MAGDM methods based on MG-3WD frameworks that can avoid impacts of the above-listed challenges. In particular, the following integration strategies can be summed up:

(1) In order to let the established models express and optimize the loss functions in the IVHF background, a new single IVHF membership degree is constructed by combining DTRSs with IVHFSs.
(2) Four kinds of IVHF MG-DTRSs over two universes are put forward according to different risk appetites of experts, i.e., the first type based on optimistic and pessimistic tactics that aims to handle extreme opinions, the second type based on the generalized IVHF hybrid averaging (GIVHFHA) operator that aims to handle groups' primary opinions, the third type based on the generalized IVHF hybrid geometric (GIVHFHG) operator that aims to handle personal primary opinions, the fourth type based on the generalized IVHF hybrid arithmetic and geometric (GIVHFHAG) operator that aims to handle both groups' primary opinions and personal primary opinions. Thereafter, the notion of multiple IVHF membership degrees is proposed.
(3) Within the framework of MG-3WD and the background of M&A target selections, some corresponding IVHF MAGDM methods are presented.

1.5. The contributions of this paper

Compared with existing IVHF MAGDM methods, the following primary contributions of the paper can be summed up:

(1) According to MG-3WD frameworks, we extend current traditional models to the background of IVHF information systems and put forward four kinds of IVHF MG-DTRSs over two universes based on different risk appetites existed in several experts, which can efficiently handle consensus processes in MAGDM problems.
(2) In light of the proposed theoretical models, general IVHF MAGDM methods are presented within the framework of MG-3WD.
(3) Case studies in the background of M&A target selections show the rationality of the presented IVHF MAGDM methods based on MG-3WD frameworks. Moreover, a sensitivity analysis together with a comparative analysis is scheduled to illustrate the availability of the established theoretical and practical models.

1.6. The structure of this paper

The remainder of the work is set out below: in Section 2, we concisely revisit several preliminary background on IVHFSs, MGRSs over two universes and DTRSs. In Section 3, we study four kinds of IVHF MG-DTRSs over two universes along with several corresponding propositions. Section 4 explores the establishment of IVHF MAGDM methods by using the proposed theoretical models. In the next section, a realistic case study with regard to a M&A target selection issue is shown to prove the correctness and validity of the constructed MAGDM approaches. Finally, several conclusions and future study options are discussed in the last section.

2. Preliminary knowledge

The following section is devoted to shortly revisiting the existing concept of IVHFSs, MGRSs over two universes and DTRSs.

2.1. IVHFSs

The notion of IVHFSs was initiated by Chen et al. [4] by substituting crisp numbers with interval numbers. With this substitution, IVHFSs can be regarded as a powerful representation tool for recording hesitant and incomplete features of various decision making preferences.

Definition 2.1 [4]**.** Let U be a finite universe of discourse, $D[0, 1]$ be the set of all closed subintervals on [0, 1]. An IVHFS $\mathbb{E}$ over U is given as a function h that when applied to U returns a subset of $D[0, 1]$, i.e.,

$$\mathbb{E} = \{\langle x, h_{\mathbb{E}}(x)\rangle | x \in U\}, \tag{1}$$

where $h_{\mathbb{E}}(x) : U \to D[0, 1]$ represents a certain number of possible membership degrees of $x \in U$ to $\mathbb{E}$. For the sake of simplicity, we name $h_{\mathbb{E}}(x)$ an IVHF element (IVHFE). In addition, a set that contains all IVHFSs on U is represented by $IVHF(U)$.

Remark 2.1. In light of Definition 2.1, $\forall \mathbb{E} \in IVHF(U)$, $\mathbb{E}$ may reduce to some special forms if the corresponding IVHFE $h_{\mathbb{E}}(x)$ satisfies some certain conditions. In specific, on the one hand, an IVHFS $\mathbb{E}$ is titled an empty IVHFS $\varnothing$ if and only if $h_{\mathbb{E}}(x) = \{[0, 0]\}$ for each $x \in U$. On the other hand, an IVHFS $\mathbb{E}$ is titled a full IVHFS $\mathbb{U}$ if and only if $h_{\mathbb{E}}(x) = \{[1, 1]\}$ for each $x \in U$.

It is significant to calculate ranking orders of diverse decision making preferences in real-world MAGDM problems. In the IVHF context, we present some common methods for comparing interval numbers and IVHFEs.

Definition 2.2 [1]**.** Suppose $a = [a^L, a^U]$ and $b = [b^L, b^U]$ are arbitrary two interval numbers. Then, the degrees of possibility $a \le b$ and $a \ge b$ are given as the following forms:

$$p(a \le b) = \max\left\{1 - \max\left(\frac{a^U - b^L}{(a^U - a^L) + (b^U - b^L)}, 0\right), 0\right\}, \tag{2}$$

$$p(a \ge b) = \max\left\{1 - \max\left(\frac{b^U - a^L}{(a^U - a^L) + (b^U - b^L)}, 0\right), 0\right\}. \tag{3}$$

Definition 2.3 [4]**.** Suppose $h_{\mathbb{E}}(x)$ is an IVHFE, $l(h_{\mathbb{E}}(x))$ is the quantity of interval numbers that are contained in $h_{\mathbb{E}}(x)$. Then, the score function of $h_{\mathbb{E}}(x)$ is given as the following form:

$$s(h_{\mathbb{E}}(x)) = \frac{\sum_{\gamma \in h_{\mathbb{E}}(x)} \gamma}{l(h_{\mathbb{E}}(x))}, \tag{4}$$

for arbitrary two IVHFEs $h_{\mathbb{E}}(x)$ and $h_{\mathbb{E}'}(x)$, $h_{\mathbb{E}}(x) \le h_{\mathbb{E}'}(x)$ satisfies if $s(h_{\mathbb{E}}(x)) \le s(h_{\mathbb{E}'}(x))$.

For an IVHFS $\mathbb{E}$, it is evident that different IVHFEs are likely to include different quantity of unsorted interval numbers, which may impede efficient operations between IVHFSs. Hence, Chen et al. [4] designed the following two viable assumptions to guarantee efficient operations between IVHFSs: (a) all interval numbers in an IVHFE $h_{\mathbb{E}}(x)$ are placed in a growing order by means of Definition 2.3. Among them, the kth largest interval number is denoted by $h_{\mathbb{E}}^{\tau(k)}(x) = [h_{\mathbb{E}}^{\tau(k)L}(x), h_{\mathbb{E}}^{\tau(k)U}(x)]$; (b) for arbitrary two IVHFEs $h_{\mathbb{E}}(x)$ and $h_{\mathbb{E}'}(x)$, if $l(h_{\mathbb{E}}(x)) \ne l(h_{\mathbb{E}'}(x))$, we shall extend the IVHFE that owns less quantity of interval numbers by supplementing the maximum interval number until $l(h_{\mathbb{E}}(x)) = l(h_{\mathbb{E}'}(x))$.

In light of the above two assumptions, the concept of IVHF subsets and several operational laws for IVHFSs are further constructed.

Definition 2.4 [42]**.** For arbitrary two IVHFEs $h_{\mathbb{E}}(x)$ and $h_{\mathbb{E}'}(x)$, $\mathbb{E}$ is named as an IVHF subset of $\mathbb{E}'$ if $h_{\mathbb{E}}(x) \preceq h_{\mathbb{E}'}(x)$ holds for each $x \in U$ such that $h_{\mathbb{E}}(x) \preceq h_{\mathbb{E}'}(x) \Leftrightarrow h_{\mathbb{E}}^{\tau(k)L}(x) \le h_{\mathbb{E}'}^{\tau(k)L}(x), h_{\mathbb{E}}^{\tau(k)U}(x) \le h_{\mathbb{E}'}^{\tau(k)U}(x)$, where $k = 1, 2, \ldots, \max\{l(h_{\mathbb{E}}(x)), l(h_{\mathbb{E}'}(x))\}$. At last, we express the above IVHF subset by $\mathbb{E} \sqsubseteq \mathbb{E}'$.

Definition 2.5 [6,42]**.** For arbitrary two IVHFSs $\mathbb{E}$ and $\mathbb{E}'$, $k = 1, 2, \ldots, \max\{l(h_{\mathbb{E}}(x)), l(h_{\mathbb{E}'}(x))\}$. Then, several operational laws for them are given as the following forms:

(1) $h_{\mathbb{E}}(x) \boxplus h_{\mathbb{E}'}(x) = \{[h_{\mathbb{E}}^{\tau(k)L}(x) + h_{\mathbb{E}'}^{\tau(k)L}(x) - h_{\mathbb{E}}^{\tau(k)L}(x)h_{\mathbb{E}'}^{\tau(k)L}(x), h_{\mathbb{E}}^{\tau(k)U}(x) + h_{\mathbb{E}'}^{\tau(k)U}(x) - h_{\mathbb{E}}^{\tau(k)U}(x)h_{\mathbb{E}'}^{\tau(k)U}(x)]\}$;

(2) $h_{\mathbb{E}}(x) \boxminus h_{\mathbb{E}'}(x) = \{[\frac{h_{\mathbb{E}}^{\tau(k)L}(x) - h_{\mathbb{E}'}^{\tau(k)L}(x)}{1 - h_{\mathbb{E}'}^{\tau(k)L}(x)}, \frac{h_{\mathbb{E}}^{\tau(k)U}(x) - h_{\mathbb{E}'}^{\tau(k)U}(x)}{1 - h_{\mathbb{E}'}^{\tau(k)U}(x)}]\}$;

(3) $h_{\mathbb{E}}(x) \boxtimes h_{\mathbb{E}'}(x) = \{[h_{\mathbb{E}}^{\tau(k)L}(x)h_{\mathbb{E}'}^{\tau(k)L}(x), h_{\mathbb{E}}^{\tau(k)U}(x)h_{\mathbb{E}'}^{\tau(k)U}(x)]\}$;

(4) $h_{\mathbb{E}}(x) \boxdot h_{\mathbb{E}'}(x) = \{[\frac{h_{\mathbb{E}}^{\tau(k)L}(x)}{h_{\mathbb{E}'}^{\tau(k)L}(x)}, \frac{h_{\mathbb{E}}^{\tau(k)U}(x)}{h_{\mathbb{E}'}^{\tau(k)U}(x)}]\}$;

(5) $\lambda(h_{\mathbb{E}}(x)) = \{[1-(1-h_{\mathbb{E}}^{\tau(k)L}(x))^{\lambda}, 1-(1-h_{\mathbb{E}}^{\tau(k)U}(x))^{\lambda}]\}$, $\lambda>0$;
(6) $(h_{\mathbb{E}}(x))^{\lambda} = \{[\left(h_{\mathbb{E}}^{\tau(k)L}(x)\right)^{\lambda}, \left(h_{\mathbb{E}}^{\tau(k)U}(x)\right)^{\lambda}]\}$, $\lambda>0$;
(7) $h_{\mathbb{E}^c}(x) = \sim h_{\mathbb{E}}(x) = \{[1-h_{\mathbb{E}}^{\tau(k)U}(x), 1-h_{\mathbb{E}}^{\tau(k)L}(x)]\}$;
(8) $h_{\mathbb{E}\Cup\mathbb{E}'}(x) = h_{\mathbb{E}}(x)\underline{\vee}h_{\mathbb{E}'}(x) = \{[h_{\mathbb{E}}^{\tau(k)L}(x)\vee h_{\mathbb{E}'}^{\tau(k)L}(x), h_{\mathbb{E}}^{\tau(k)U}(x)\vee h_{\mathbb{E}'}^{\tau(k)U}(x)]\}$;
(9) $h_{\mathbb{E}\Cap\mathbb{E}'}(x) = h_{\mathbb{E}}(x)\overline{\wedge}h_{\mathbb{E}'}(x) = \{[h_{\mathbb{E}}^{\tau(k)L}(x)\wedge h_{\mathbb{E}'}^{\tau(k)L}(x), h_{\mathbb{E}}^{\tau(k)U}(x)\wedge h_{\mathbb{E}'}^{\tau(k)U}(x)]\}$.

Definition 2.6 [4]**.** For a series of IVHFEs $h_{\mathbb{E}_i}(x)$ $(i=1,2,\ldots,m)$, $\boldsymbol{\omega}=(\omega_1,\omega_2,\ldots,\omega_m)^T$ is the weight vector such that $\omega_i\in[0,1]$ and $\sum_{i=1}^{m}\omega_i=1$. Then, the GIVHFHA operator and the GIVHFHG operator are given as the following forms:

(1) The GIVHFHA operator:

$$GIVHFHA_{\lambda}(h_{\mathbb{E}_1}(x), h_{\mathbb{E}_2}(x), \ldots, h_{\mathbb{E}_m}(x)) = \left\{\left[\left(1-\prod_{i=1}^{m}\left(1-\left(h_{\mathbb{E}_i}^{\tau(i)L}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}, \left(1-\prod_{i=1}^{m}\left(1-\left(h_{\mathbb{E}_i}^{\tau(i)U}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right]\right\}, \lambda>0. \tag{5}$$

(2) The GIVHFHG operator:

$$GIVHFHG_{\lambda}(h_{\mathbb{E}_1}(x), h_{\mathbb{E}_2}(x), \ldots, h_{\mathbb{E}_m}(x)) = \left\{\left[1-\left(1-\prod_{i=1}^{m}\left(1-\left(1-h_{\mathbb{E}_i}^{\tau(i)L}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}, 1-\left(1-\prod_{i=1}^{m}\left(1-\left(1-h_{\mathbb{E}_i}^{\tau(i)U}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right]\right\}, \lambda>0. \tag{6}$$

In the case when $\boldsymbol{\omega}=\left(\frac{1}{m},\frac{1}{m},\ldots,\frac{1}{m}\right)^T$ holds, the GIVHFHA operator reduces to the IVHF ordered weighted averaging (GIVHFOWA) operator, and the GIVHFHG operator reduces to the IVHF ordered weighted geometric (GIVHFOWG) operator. In addition, in the case when $\lambda=1$, the GIVHFHA operator reduces to the IVHF hybrid averaging (IVHFHA) operator, and the GIVHFHG operator reduces to the IVHF hybrid geometric (IVHFHG) operator. It is noted that the GIVHFOWA operator and the GIVHFOWG operator only weight the ordered position of each item, but ignore the significance of the same item, and the IVHFHA operator and the IVHFHG operator fail to consider all the given items and their ordered positions, which may affect the aggregation results in the IVHF background. Thus, the GIVHFHA operator and the GIVHFHG operator outweigh the above-mentioned counterparts in the IVHF information fusion.

2.2. MGRSs over two universes

By referring to a two-universe framework, the concept of MGRSs over two universes [23] was initiated. Compared with traditional rough set models, MGRSs over two universes act as a reasonable way to address multi-strategy and multi-level complicated MAGDM problems.

Definition 2.7 [23]**.** Suppose U and V are two finite universes of discourse, $\mathcal{R}$ denotes a family of binary compatibility relations from U to V with respect to a binary mapping family F_i, where F_i: $U\to 2^V$, $u\mapsto\{(u,v)\in R_i|v\in V.\}$, $R_i\in\mathcal{R}$, $i=1,2,\ldots,m$. Afterwards, suppose A and A' are two mappings stated above, for each $X\subseteq V$, the optimistic and pessimistic multigranulation approximations of X are given as the following forms:

$$\underline{approx}_{A+A'}^{O}(X) = \{A(x)\subseteq X \vee A'(x)\subseteq X\}; \tag{7}$$

$$\overline{approx}_{A+A'}^{O}(X) = \underline{approx}_{A+A'}^{O}(X^c)^c; \tag{8}$$

$$\underline{approx}_{A+A'}^{P}(X) = \{A(x)\subseteq X \wedge A'(x)\subseteq X\}; \tag{9}$$

$$\overline{approx}_{A+A'}^{P}(X) = \underline{approx}_{A+A'}^{P}(X^c)^c, \tag{10}$$

the pair $(\underline{approx}_{A+A'}^{O}(X), \overline{approx}_{A+A'}^{O}(X))$ is titled an optimistic MGRS over two universes of X, whereas the pair $(\underline{approx}_{A+A'}^{P}(X), \overline{approx}_{A+A'}^{P}(X))$ is titled a pessimistic MGRS over two universes of X.

2.3. DTRSs

The model of DTRSs, proposed and developed by Yao [37], is one popular way to realize general ideas of 3WD based on the integration result of Bayesian decision procedures and rough sets. In the procedure of constructing a DTRS, two states $\Omega = \{G, G^c\}$ are constructed at first, which represent an element is in G or not in G. Afterwards, three actions $\mathcal{A} = \{a_P, a_B, a_N\}$ are constructed, which represent the actions of classifications for an element x. Thereafter, λ_{PP}, λ_{BP} and λ_{NP} represent the losses triggered for choosing a_P, a_B and a_N when $x \in G$, whereas λ_{PN}, λ_{BN} and λ_{NN} represent the losses triggered for choosing the identical actions when $x \in G^c$. Then, the estimated costs of choosing diverse actions for x are given as the following forms:

$$\mathcal{L}(a_P|[x]) = \lambda_{PP}P(G|[x]) + \lambda_{PN}P(G^c|[x]); \tag{11}$$

$$\mathcal{L}(a_B|[x]) = \lambda_{BP}P(G|[x]) + \lambda_{BN}P(G^c|[x]); \tag{12}$$

$$\mathcal{L}(a_N|[x]) = \lambda_{NP}P(G|[x]) + \lambda_{NN}P(G^c|[x]). \tag{13}$$

In light of Bayesian decision procedures, the minimum-risk decision rules can be reached below. (P) If $x \in POS(G)$ is reached such that $\mathcal{L}(a_P|[x]) < \mathcal{L}(a_B|[x])$ and $\mathcal{L}(a_P|[x]) < \mathcal{L}(a_N|[x])$ hold.

(B) If $x \in BND(G)$ is reached such that $\mathcal{L}(a_B|[x]) < \mathcal{L}(a_P|[x])$ and $\mathcal{L}(a_B|[x]) < \mathcal{L}(a_N|[x])$ hold.

(N) If $x \in NEG(G)$ is reached such that $\mathcal{L}(a_N|[x]) < \mathcal{L}(a_P|[x])$ and $\mathcal{L}(a_N|[x]) < \mathcal{L}(a_B|[x])$ hold.

Taking into account loss functions with $\lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$ and $\lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$ usually hold in plenty of realistic decision making situations. Moreover, if we have the following requirements, i.e., $(\lambda_{PN} - \lambda_{BN})(\lambda_{NP} - \lambda_{BP}) \ge (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$, $P(G|[x]) + P(G^c|[x]) = 1$ and $\alpha > \beta$, then a series of relatively simplified decision rules can be further reached with $\alpha = \frac{(\lambda_{PN}-\lambda_{BN})}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})}$ and $\beta = \frac{(\lambda_{BN}-\lambda_{NN})}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})}$. (P_1) If $x \in POS(G)$ is reached such that $P(G|[x].) \ge \alpha$ holds.

(B_1) If $x \in BND(G)$ is reached such that $\beta < P(G|[x].) < \alpha$ holds.

(N_1) If $x \in NEG(G)$ is reached such that $P(G|[x].) \le \beta$ holds.

3. IVHF MG-DTRSs over two universes

Considering various merits of IVHF information systems, we aim to explore several reasonable models by fusing DTRSs with MGRSs over two universes in this section. The motivation of developing several models lies in how to cope with consensus and conflicting opinions provided by different decision makers in a typical group decision making process. In specific, there often exist four common group decision making situations: (i) all decision makers hold extreme opinions (completely risk-seeking or completely risk-averse) for group decision making situations. In this case, we aim to develop type-I IVHF MG-DTRSs over two universes based on optimistic and pessimistic tactics; (ii) decision makers hold conflicting opinions for group decision making situations, and they agree to adopt groups' primary opinions. In this case, we aim to develop type-II IVHF MG-DTRSs over two universes based on GIVHFHA operators; (iii) decision makers hold conflicting opinions for group decision making situations, and they agree to adopt personal primary opinions. In this case, we aim to develop type-III IVHF MG-DTRSs over two universes based on GIVHFHG operators; (iv) decision makers hold conflicting opinions for group decision making situations, and they agree to adopt both groups' primary opinions and personal primary opinions. In this case, we aim to develop type-IV IVHF MG-DTRSs over two universes based on GIVHFHAG operators. For each theoretical model, both fundamental definitions and primary propositions are investigated in detail.

3.1. Type-I IVHF MG-DTRSs over two universes

In accordance with pioneering papers with respect to MGRSs models, the strategy of extending single granulation rough sets (SGRSs) to MGRSs is two-fold, i.e., the optimistic one and the pessimistic one. Concretely, the term "optimistic" refers to at least one granular structure should be employed for satisfying the requirement of the inclusion prerequisite between approximated objects and equivalence classes, whereas the term "pessimistic" refers to all granular structures ought to be considered for completing the identical task. Inspired by the above-stated two different tactics, this section plans to study optimistic and pessimistic MGRSs over two universes by means of IVHFSs and DTRSs. In order to promote the development of the presented models, the notion of IVHF relations over two universes and single IVHF membership degrees are explored at first.

Definition 3.1 [42]. Let $\mathbb{R}$ be an IVHF relation over $U \times V$, which is expressed by an IVHFS. Then, $\mathbb{R}$ is given as the following form:

$$\mathbb{R} = \{\langle (x, y), h_{\mathbb{R}}(x, y)\rangle | (x, y) \in U \times V\}, \tag{14}$$

where $h_{\mathbb{R}}(x, y) : U \times V \to D[0, 1]$ denotes the possible membership degrees of the pair (x, y) to $U \times V$. Moreover, a set that contains all IVHF relations over two universes is represented by $IVHFR(U \times V)$.

Definition 3.2. Let $\mathbb{R}$ be an IVHF relation over $U \times V$. For any $\mathbb{E} \in IVHF(V)$, $x \in U$ and $y \in V$, the single IVHF membership degree of x in $\mathbb{E}$ in terms of $\mathbb{R}$, represented by $\Theta_{\mathbb{E}}^{\mathbb{R}}(x)$, is given as the following form:

$$\Theta_{\mathbb{E}}^{\mathbb{R}}(x) = \frac{\sum_{y \in V} \mathbb{E}(y)\mathbb{R}(x,y)}{\mathbb{R}(x,y)}. \tag{15}$$

In what follows, we generalize single IVHF membership degrees to the multigranulation background, and further put forward two forms of multiple IVHF membership degrees.

Definition 3.3. Given an IVHF relation over two universes $\mathbb{R}_i \in IVHFR(U \times V)(i = 1, 2, \ldots, m)$. For any $\mathbb{E} \in IVHF(V)$, $x \in U$ and $y \in V$, the maximal version and the minimal version concerning multiple IVHF membership degrees of x in $\mathbb{E}$ in terms of $\mathbb{R}_i$, represented by $\Omega_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x)$ and $\Psi_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x)$, are given as the following form:

$$\Omega_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x) = \max_{i=1}^m \Theta_{\mathbb{E}}^{\mathbb{R}_i}(x) = \max_{i=1}^m \left\{ \frac{\sum_{y \in V} \mathbb{E}(y)\mathbb{R}_i(x,y)}{\mathbb{R}_i(x,y)} \right\}; \tag{16}$$

$$\Psi_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x) = \min_{i=1}^m \Theta_{\mathbb{E}}^{\mathbb{R}_i}(x) = \min_{i=1}^m \left\{ \frac{\sum_{y \in V} \mathbb{E}(y)\mathbb{R}_i(x,y)}{\mathbb{R}_i(x,y)} \right\}. \tag{17}$$

Following the form of maximal and minimal multiple IVHF membership degrees, we further construct two versions of type-I IVHF MG-DTRSs over two universes.

Definition 3.4. Suppose the threshold parameters α and β are two IVHFEs. For any $\mathbb{E} \in IVHF(V)$, $x \in U$, $y \in V$ and $\beta < \alpha$, the optimistic IVHF multigranulation decision-theoretic (MG-DT) rough approximations of $\mathbb{E}$ with regard to $\mathbb{R}_i$ are represented by $\underline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E})$ and $\overline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\beta}(\mathbb{E})$, whereas the pessimistic IVHF MG-DT rough approximations of $\mathbb{E}$ with regard to $\mathbb{R}_i$ are represented by $\underline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E})$ and $\overline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\beta}(\mathbb{E})$. Then, the above-stated four rough approximations are given as the following form:

$$\underline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) = \left\{ \Omega_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x) \geq \alpha | x \in U \right\}; \tag{18}$$

$$\overline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}) = \left\{ \Omega_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x) > \beta | x \in U \right\}; \tag{19}$$

$$\underline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E}) = \left\{ \Psi_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x) \geq \alpha | x \in U \right\}; \tag{20}$$

$$\overline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\beta}(\mathbb{E}) = \left\{ \Psi_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x) > \beta | x \in U \right\}, \tag{21}$$

the pair $(\underline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}), \overline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}))$ is titled an optimistic IVHF MG-DTRSs over two universes, whereas the pair $(\underline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E}), \overline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\beta}(\mathbb{E}))$ is titled a pessimistic IVHF MG-DTRSs over two universes. In addition, the corresponding positive, negative and boundary regions with respect to optimistic IVHF MG-DT rough approximations are listed as follows:

(1) $POS_{\alpha,\beta}^{\Omega}(\mathbb{E}) = \underline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E})$;
(2) $NEG_{\alpha,\beta}^{\Omega}(\mathbb{E}) = U - \overline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\beta}(\mathbb{E})$;
(3) $BND_{\alpha,\beta}^{\Omega}(\mathbb{E}) = \overline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}) - \underline{\sum_{i=1}^m \mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E})$.

Afterwards, the three regions with respect to pessimistic IVHF MG-DT rough approximations are listed as follows:

(1) $POS_{\alpha,\beta}^{\Psi}(\mathbb{E}) = \underline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E})$;
(2) $NEG_{\alpha,\beta}^{\Psi}(\mathbb{E}) = U - \overline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\beta}(\mathbb{E})$;
(3) $BND_{\alpha,\beta}^{\Psi}(\mathbb{E}) = \overline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\beta}(\mathbb{E}) - \underline{\sum_{i=1}^m \mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E})$.

In the following, some key propositions of optimistic IVHF MG-DT rough approximations are explored.

Proposition 3.1. *Given an IVHF relation over two universes $\mathbb{R}_i \in IVHFR(U \times V)(i = 1, 2, \ldots, m)$, and threshold parameters α and β are expressed by IVHFEs. Then the optimistic IVHF MG-DT rough approximations of $\mathbb{E}$ own the following properties:*

(1) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E})$;

(2) $\mathbb{E} \subseteq \mathbb{E}' \Leftrightarrow \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}')$, $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}')$;

(3) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E} \Cup \mathbb{E}') \sqsupseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) \Cup \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}')$, $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E} \Cap \mathbb{E}') \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}) \Cap \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}')$;

(4) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E} \Cap \mathbb{E}') \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) \Cap \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}')$, $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E} \Cup \mathbb{E}') \sqsupseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}) \Cup \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}')$;

(5) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\varnothing) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\varnothing) = \varnothing$, $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{U}) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{U}) = \mathbb{U}$.

Proof.

(1) In light of Definition 3.4, since $\beta < \alpha$, we have

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) = \left\{ \Omega_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) \geq \alpha | x \in U \right\} \subseteq$$
$$\left\{ \Omega_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) > \beta | x \in U \right\} = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}).$$

Thus, $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E})$ can be obtained.

(2) According to Definitions 3.3 and 3.4, since $\mathbb{E} \sqsubseteq \mathbb{E}'$, we have

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) = \left\{ \Omega_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) \geq \alpha | x \in U \right\} = \left\{ \max_{i=1}^{m} \Theta_{\mathbb{E}}^{\mathbb{R}_i}(x) \geq \alpha | x \in U \right\}$$
$$= \left\{ \max_{i=1}^{m} \left\{ \frac{\sum_{y \in V} \mathbb{E}(y)\mathbb{R}_i(x,y)}{\mathbb{R}_i(x,y)} \right\} \geq \alpha | x \in U \right\} \leq \left\{ \max_{i=1}^{m} \left\{ \frac{\sum_{y \in V} \mathbb{E}'(y)\mathbb{R}_i(x,y)}{\mathbb{R}_i(x,y)} \right\} \geq \alpha | x \in U \right\}$$
$$= \left\{ \max_{i=1}^{m} \Theta_{\mathbb{E}'}^{\mathbb{R}_i}(x) \geq \alpha | x \in U \right\} = \left\{ \Omega_{\mathbb{E}'}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) \geq \alpha | x \in U \right\} = \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}').$$

Thus, $\mathbb{E} \sqsubseteq \mathbb{E}' \Leftrightarrow \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}')$ can be obtained. In a similar manner, we can also obtain $\mathbb{E} \sqsubseteq \mathbb{E}' \Leftrightarrow \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}')$.

(3) According to Definitions 3.3 and 3.4, we have

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E} \Cup \mathbb{E}') = \left\{ \Omega_{\mathbb{E} \Cup \mathbb{E}'}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) \geq \alpha | x \in U \right\} = \left\{ \max_{i=1}^{m} \Theta_{\mathbb{E} \cup \mathbb{E}'}^{\mathbb{R}_i}(x) \geq \alpha | x \in U \right\}$$
$$= \left\{ \max_{i=1}^{m} \left\{ \frac{\sum_{y \in V} (\mathbb{E} \cup \mathbb{E}')(y)\mathbb{R}_i(x,y)}{\mathbb{R}_i(x,y)} \right\} \geq \alpha | x \in U \right\} \geq \left\{ \max_{i=1}^{m} \left\{ \frac{\sum_{y \in V} \mathbb{E}(y)\mathbb{R}_i(x,y)}{\mathbb{R}_i(x,y)} \right\} \geq \alpha | x \in U \right\}.$$

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E} \Cup \mathbb{E}') = \left\{ \Omega_{\mathbb{E} \Cup \mathbb{E}'}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) \geq \alpha | x \in U \right\} = \left\{ \max_{i=1}^{m} \Theta_{\mathbb{E} \cup \mathbb{E}'}^{\mathbb{R}_i}(x) \geq \alpha | x \in U \right\}$$
$$= \left\{ \max_{i=1}^{m} \left\{ \frac{\sum_{y \in V} (\mathbb{E} \cup \mathbb{E}')(y)\mathbb{R}_i(x,y)}{\mathbb{R}_i(x,y)} \right\} \geq \alpha | x \in U \right\} \geq \left\{ \max_{i=1}^{m} \left\{ \frac{\sum_{y \in V} \mathbb{E}'(y)\mathbb{R}_i(x,y)}{\mathbb{R}_i(x,y)} \right\} \geq \alpha | x \in U \right\}.$$

Thus, $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E} \Cup \mathbb{E}') \sqsupseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) \Cup \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}')$ can be obtained. In a similar manner, we can also obtain $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E} \Cap \mathbb{E}') \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}) \Cap \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}')$.

(4) In light of the above results, $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E} \Cap \mathbb{E}') \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}) \Cap \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{E}')$ and $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E} \Cup \mathbb{E}') \sqsupseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}) \Cup \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{E}')$ can be obtained.

(5) According to Definitions 3.3 and 3.4, we have

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\varnothing)=\left\{\Omega_{\varnothing}^{\sum_{i=1}^{m}\mathbb{R}_i}(x)\geq\alpha\,|\,x\in U\right\}$$

$$=\left\{\max_{i=1}^{m}\Theta_{\varnothing}^{\mathbb{R}_i}(x)\geq\alpha\,|\,x\in U\right\}=\left\{\max_{i=1}^{m}\left\{\frac{\sum_{y\in V}\{[0,0]\}\mathbb{R}_i(x,y)}{\mathbb{R}_i(x,y)}\right\}\geq\alpha\,|\,x\in U\right\}=\varnothing.$$

Thus, $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\varnothing)=\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\varnothing)=\varnothing$ can be obtained. In a similar manner, we can also obtain $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\alpha}(\mathbb{U})=\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Omega,\beta}(\mathbb{U})=\mathbb{U}$. □

Similarly, some corresponding key propositions of pessimistic IVHF MG-DT rough approximations can be listed below, and it is not difficult to prove them based on proofs of Proposition 3.1.

Proposition 3.2. *Given an IVHF relation over two universes $\mathbb{R}_i\in IVHFR(U\times V)(i=1,2,\ldots,m)$, and threshold parameters α and β are expressed by IVHFEs. Then the pessimistic IVHF MG-DT rough approximations of $\mathbb{E}$ own the following properties:*

(1) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E})\sqsubseteq\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{E})$;

(2) $\mathbb{E}\subseteq\mathbb{E}'\Leftrightarrow\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E})\sqsubseteq\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E}')$, $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{E})\sqsubseteq\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{E}')$;

(3) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E}\Cup\mathbb{E}')\sqsupseteq\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E})\Cup\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E}')$, $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{E}\Cap\mathbb{E}')\sqsubseteq\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{E})\Cap\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{E}')$;

(4) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E}\Cap\mathbb{E}')\sqsubseteq\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E})\Cap\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{E}')$, $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{E}\Cup\mathbb{E}')\sqsupseteq\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{E})\Cup\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{E}')$;

(5) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\varnothing)=\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\varnothing)=\varnothing$, $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\alpha}(\mathbb{U})=\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Psi,\beta}(\mathbb{U})=\mathbb{U}$.

3.2. Type-II IVHF MG-DTRSs over two universes

In accordance with the above-stated information fusion tactics (the optimistic one and the pessimistic one), the proposed model titled type-I IVHF MG-DTRSs over two universes consist of different versions. Moreover, in view of the concrete forms of type-I IVHF MG-DTRSs over two universes, the optimistic tactic uses maximum operators to construct related rough approximations, which seeks common ground and reserves differences on this issue. On the contrary, the pessimistic tactic uses minimum operators to construct related rough approximations, which seeks common ground and rejects differences on this issue. Hence, absoluteness and extremeness can be found in both optimistic and pessimistic tactics. In order to overcome the above-mentioned limitations, establishing viable multiple IVHF membership degrees by means of maximal and minimal multiple IVHF membership degrees is necessary. In the following, we aim to present a novel multiple IVHF membership degree based on GIVHFHA operators, then the definitions and propositions of type-II IVHF MG-DTRSs over two universes are explored.

Next, we put forward a new multiple IVHF membership degree in light of GIVHFHA operators.

Definition 3.5. Given an IVHF relation over two universes $\mathbb{R}_i\in IVHFR(U\times V)(i=1,2,\ldots,m)$, ω_i is the weight of $\mathbb{R}_i$. For any $\mathbb{E}\in IVHF(V)$, then the concept of multiple IVHF membership degrees by virtue of GIVHFHA operators is given as the following form:

$$\Xi_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x)=GIVHFHA_{\lambda}\left(\Theta_{\mathbb{E}}^{\mathbb{R}_1}(x),\Theta_{\mathbb{E}}^{\mathbb{R}_2}(x),\ldots,\Theta_{\mathbb{E}}^{\mathbb{R}_m}(x)\right)$$
$$=\left\{\left[\left(1-\prod_{i=1}^{m}\left(1-\left(\Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}L}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda},\left(1-\prod_{i=1}^{m}\left(1-\left(\Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}U}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right]\right\},\lambda>0.\tag{22}$$

In light of the notion of multiple IVHF membership degrees by virtue of GIVHFHA operators, type-II IVHF MG-DTRSs over two universes are constructed in the following definition.

Definition 3.6. Suppose the threshold parameters α and β are two IVHFEs. For any $\mathbb{E}\in IVHF(V)$, $x\in U$, $y\in V$ and $\beta<\alpha$, the type-II IVHF MG-DT rough approximations of $\mathbb{E}$ with regard to $\mathbb{R}_i$ are represented by $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E})$ and $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E})$. Then, the above-stated two rough approximations are given as the following form:

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E})=\left\{\Xi_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x)\geq\alpha\,|\,x\in U\right\};\tag{23}$$

$$\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E}) = \left\{ \Xi_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) > \beta | x \in U \right\}, \tag{24}$$

the pair $(\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E}), \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E}))$ is titled a type-II IVHF MG-DTRSs over two universes. In addition, the corresponding positive, negative and boundary regions with respect to type-II IVHF MG-DT rough approximations are listed as follows:

(1) $POS_{\alpha,\beta}^{\Xi}(\mathbb{E}) = \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E})$;
(2) $NEG_{\alpha,\beta}^{\Xi}(\mathbb{E}) = U - \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E})$;
(3) $BND_{\alpha,\beta}^{\Xi}(\mathbb{E}) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E}) - \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E})$.

Similarly, some corresponding key propositions of type-II IVHF MG-DT rough approximations can be listed below, and it is not difficult to prove them based on proofs of Proposition 3.1.

Proposition 3.3. *Given an IVHF relation over two universes $\mathbb{R}_i \in IVHFR(U \times V)(i = 1, 2, \ldots, m)$, and threshold parameters α and β are expressed by IVHFEs. Then the type-II IVHF MG-DT rough approximations of $\mathbb{E}$ own the following properties:*

(1) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E})$;
(2) $\mathbb{E} \subseteq \mathbb{E}' \Leftrightarrow \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E}) \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E}'),\ \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E}')$;
(3) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E} \Cup \mathbb{E}') \sqsupseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E}) \Cup \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E}'),\ \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E} \Cap \mathbb{E}') \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E}) \Cap \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E}')$;
(4) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E} \Cap \mathbb{E}') \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E}) \Cap \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{E}'),\ \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E} \Cup \mathbb{E}') \sqsupseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E}) \Cup \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{E}')$;
(5) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\varnothing) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\varnothing) = \varnothing,\ \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\alpha}(\mathbb{U}) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Xi,\beta}(\mathbb{U}) = \mathbb{U}$.

3.3. Type-III IVHF MG-DTRSs over two universes

Since GIVHFHG operators provide another reasonable way to integrate multiple IVHF membership degrees, it is meaningful to present the concept of multiple IVHF membership degrees by virtue of GIVHFHG operators, then type-III IVHF MG-DTRSs over two universes are put forward below.

Definition 3.7. Given an IVHF relation over two universes $\mathbb{R}_i \in IVHFR(U \times V)(i = 1, 2, \ldots, m)$, ω_i is the weight of $\mathbb{R}_i$. For any $\mathbb{E} \in IVHF(V)$, then the concept of multiple IVHF membership degrees by virtue of GIVHFHG operators is given as the following form:

$$\begin{aligned} Z_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) &= GIVHFHG_{\lambda}(\Theta_{\mathbb{E}}^{\mathbb{R}_1}(x), \Theta_{\mathbb{E}}^{\mathbb{R}_2}(x), \ldots, \Theta_{\mathbb{E}}^{\mathbb{R}_m}(x)) \\ &= \left\{ \left[1 - \left(1 - \prod_{i=1}^{m} \left(1 - \left(1 - \Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}L}(x) \right)^{\lambda} \right)^{\omega_i} \right)^{1/\lambda}, 1 - \left(1 - \prod_{i=1}^{m} \left(1 - \left(1 - \Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}U}(x) \right)^{\lambda} \right)^{\omega_i} \right)^{1/\lambda} \right] \right\}, \lambda > 0. \end{aligned} \tag{25}$$

In accordance with the concept of multiple IVHF membership degrees by virtue of GIVHFHG operators, type-III IVHF MG-DTRSs over two universes are constructed in the following definition.

Definition 3.8. Suppose the threshold parameters α and β are two IVHFEs. For any $\mathbb{E} \in IVHF(V)$, $x \in U$, $y \in V$ and $\beta < \alpha$, the type-III IVHF MG-DT rough approximations of $\mathbb{E}$ with regard to $\mathbb{R}_i$ are represented by $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{E})$ and $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{E})$. Then, the above-stated two rough approximations are given as the following form:

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{E}) = \left\{ Z_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) \geq \alpha | x \in U \right\}; \tag{26}$$

$$\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{E}) = \left\{ Z_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) > \beta | x \in U \right\}, \tag{27}$$

the pair $(\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{E}), \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{E}))$ is titled a type-III IVHF MG-DTRSs over two universes. In addition, the corresponding positive, negative and boundary regions with respect to type-III IVHF MG-DT rough approximations are listed as follows:

(1) $POS^{Z}_{\alpha,\beta}(\mathbb{E}) = \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{E})$;
(2) $NEG^{Z}_{\alpha,\beta}(\mathbb{E}) = U - \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{E})$;
(3) $BND^{Z}_{\alpha,\beta}(\mathbb{E}) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{E}) - \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{E})$.

Similarly, some corresponding key propositions of type-III IVHF MG-DT rough approximations can be listed below, and it is not difficult to prove them based on proofs of Proposition 3.1.

Proposition 3.4. *Given an IVHF relation over two universes $\mathbb{R}_i \in IVHFR(U \times V)(i = 1, 2, \ldots, m)$, and threshold parameters α and β are expressed by IVHFEs. Then the type-III IVHF MG-DT rough approximations of $\mathbb{E}$ own the following properties:*

(1) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{E})$;
(2) $\mathbb{E} \subseteq \mathbb{E}' \Leftrightarrow \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{E}) \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}\left(\mathbb{E}'\right),\ \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}\left(\mathbb{E}'\right)$;
(3) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}\left(\mathbb{E} \Cup \mathbb{E}'\right) \sqsupseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{E}) \Cup \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}\left(\mathbb{E}'\right),\ \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}\left(\mathbb{E} \Cap \mathbb{E}'\right) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{E}) \Cap \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}\left(\mathbb{E}'\right)$;
(4) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}\left(\mathbb{E} \Cap \mathbb{E}'\right) \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{E}) \Cap \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}\left(\mathbb{E}'\right),\ \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}\left(\mathbb{E} \Cup \mathbb{E}'\right) \sqsupseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{E}) \Cup \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}\left(\mathbb{E}'\right)$;
(5) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\varnothing) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\varnothing) = \varnothing,\ \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\alpha}(\mathbb{U}) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{Z,\beta}(\mathbb{U}) = \mathbb{U}$.

3.4. Type-IV IVHF MG-DTRSs over two universes

In light of classical fuzzy aggregation operators, it is worth noticing that GIVHFHA operators tend to select groups' primary opinions, whereas GIVHFHG operators lay stress on personal primary opinions. In order to simultaneously overcome the limitations of GIVHFHA operators and GIVHFHG operators in several practical cases, establishing the notion of GIVHFHAG operators by means of GIVHFHA operators and GIVHFHG operators is necessary. In the following, the concept of multiple IVHF membership degrees by virtue of GIVHFHAG operators are developed, then type-IV IVHF MG-DTRSs over two universes are further investigated.

Definition 3.9. Given an IVHF relation over two universes $\mathbb{R}_i \in IVHFR(U \times V)(i = 1, 2, \ldots, m)$, ω_i is the weight of $\mathbb{R}_i$. For any $\mathbb{E} \in IVHF(V)$, then the concept of multiple IVHF membership degrees by virtue of GIVHFHAG operators is given as the following form:

$$\Phi_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) = GIVHFHAG_{\lambda}\left(\Theta_{\mathbb{E}}^{\mathbb{R}_1}(x), \Theta_{\mathbb{E}}^{\mathbb{R}_2}(x), \ldots, \Theta_{\mathbb{E}}^{\mathbb{R}_m}(x)\right)$$
$$= \left\{\left[\left(1 - \prod_{i=1}^{m}\left(1 - \left(\Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}L}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}, \left(1 - \prod_{i=1}^{m}\left(1 - \left(\Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}U}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right]\right\}^{1/2}$$
$$\boxtimes \left\{\left[1 - \left(1 - \prod_{i=1}^{m}\left(1 - \left(1 - \Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}L}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}, 1 - \left(1 - \prod_{i=1}^{m}\left(1 - \left(1 - \Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}U}(x)\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right]\right\}^{1/2}, \lambda > 0. \tag{28}$$

Based on the notion of multiple IVHF membership degrees by virtue of GIVHFHAG operators, type-IV IVHF MG-DTRSs over two universes are constructed in the following definition.

Definition 3.10. Suppose the threshold parameters α and β are two IVHFEs. For any $\mathbb{E} \in IVHF(V)$, $x \in U$, $y \in V$ and $\beta < \alpha$, the type-IV IVHF MG-DT rough approximations of $\mathbb{E}$ with regard to $\mathbb{R}_i$ are represented by $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{E})$ and $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{E})$. Then, the above-stated two rough approximations are given as the following form:

$$\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{E}) = \left\{\Phi_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) \geq \alpha | x \in U\right\}; \tag{29}$$

$$\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{E}) = \left\{\Phi_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x) > \beta | x \in U\right\}, \tag{30}$$

the pair $(\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{E}), \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{E}))$ is titled a type-IV IVHF MG-DTRSs over two universes. In addition, the corresponding positive, negative and boundary regions with respect to type-IV IVHF MG-DT rough approximations are listed as follows:

(1) $POS^{\Phi}_{\alpha,\beta}(\mathbb{E}) = \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{E})$;

(2) $NEG^{\Phi}_{\alpha,\beta}(\mathbb{E}) = U - \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{E})$;

(3) $BND^{\Phi}_{\alpha,\beta}(\mathbb{E}) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{E}) - \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{E})$.

Similarly, some corresponding key propositions of type-IV IVHF MG-DT rough approximations can be listed below, and it is not difficult to prove them based on proofs of Proposition 3.1.

Proposition 3.5. *Given an IVHF relation over two universes $\mathbb{R}_i \in IVHFR(U \times V)(i = 1, 2, \ldots, m)$, and threshold parameters α and β are expressed by IVHFEs. Then the type-IV IVHF MG-DT rough approximations of $\mathbb{E}$ own the following properties:*

(1) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{E})$;

(2) $\mathbb{E} \subseteq \mathbb{E}' \Leftrightarrow \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{E}) \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}\left(\mathbb{E}'\right)$, $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{E}) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}\left(\mathbb{E}'\right)$;

(3) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}\left(\mathbb{E} \Cup \mathbb{E}'\right) \sqsupseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{E}) \Cup \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}\left(\mathbb{E}'\right)$, $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}\left(\mathbb{E} \Cap \mathbb{E}'\right) \sqsubseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{E}) \Cap \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}\left(\mathbb{E}'\right)$;

(4) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}\left(\mathbb{E} \Cap \mathbb{E}'\right) \sqsubseteq \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{E}) \Cap \underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}\left(\mathbb{E}'\right)$, $\overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}\left(\mathbb{E} \Cup \mathbb{E}'\right) \sqsupseteq \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{E}) \Cup \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}\left(\mathbb{E}'\right)$;

(5) $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\varnothing) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\varnothing) = \varnothing$, $\underline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\alpha}(\mathbb{U}) = \overline{\sum_{i=1}^{m}\mathbb{R}_i}^{\Phi,\beta}(\mathbb{U}) = \mathbb{U}$.

4. MAGDM based on IVHF MG-DTRSs over two universes

The following part aims to explore comprehensive MAGDM approaches by virtue of the proposed IVHF MG-DTRSs over two universes. Additionally, the corresponding constructed MAGDM approaches in the background of M&A target selections are scheduled to investigate. M&A target selection is a promising and hot research topic in the discipline of economics and management. With the support of intelligent M&A target selection rules, business organizations can effectively grow or downsize, and change the nature of their business or market positioning, thus it is necessary to study reasonable mathematical models of M&A target selection methods for various demanding enterprises.

4.1. Problem statement

As a significant commercial activity in the era of business intelligence, M&A generally means consolidation of different enterprises or assets via diverse types of financial transactions. A typical M&A process usually involves a series of different financial transactions, including mergers, acquisitions, purchase of assets, tender offers, consolidations and management acquisitions. Before establishing a new firm by consolidations, in order to achieve maximum benefits and realize a sustainable development, the boards of directors of acquiring companies need to determine the optimal acquired companies by analyzing massive M&A data, hence M&A target selections have a far-reaching influence on the whole M&A procedures.

In recent years, owing to the unique merits when expressing various kinds of uncertain information in the M&A target selection background, several works have been conducted by virtue of diverse generalized fuzzy approaches. Consequently, we plan to transform M&A target selection problems into IVHF MAGDM problems, and then study corresponding problem solving methods by means of the proposed four kinds of theoretical models.

4.2. Application model

Prior to the exploration of MAGDM based on the proposed four kinds of theoretical models, several steps of transforming M&A target selection problems into IVHF MAGDM problems are presented at first. Considering relationships between optional M&A targets and assessment factors play a primary role in M&A target selection problems, we suppose $U = \{x_1, x_2, \ldots, x_p\}$ is a set of optional M&A targets, and $V = \{y_1, y_2, \ldots, y_q\}$ is a set of assessment factors. Then, each member from board of directors of an acquiring company makes a decision for the relationship between optional M&A targets and assessment factors, which is denoted by $\mathbb{R}_i \in IVHFR(U \times V)(i = 1, 2, \ldots, m)$. Afterwards, the same board of directors of an acquiring company also confirms a standard set $\mathbb{E} \in IVHF(V)$ that represents the requirements of acquiring companies by means of assessment factors. Finally, an information system of M&A target selections $(U, V, \mathbb{R}_i, \mathbb{E})$ is constructed for further selecting the optimal M&A target.

In light of an information system of M&A target selections $(U, V, \mathbb{R}_i, \mathbb{E})$, general MAGDM approaches are presented by using IVHF MG-DTRSs over two universes, and we further represent final conclusions as optimal M&A targets, excluded M&A targets, and M&A targets need additional evaluation. Hence, the MAGDM procedure can be regarded as deciding the corresponding positive, negative and boundary regions in terms of $(U, V, \mathbb{R}_i, \mathbb{E})$.

Based on the model of DTRSs presented in Section 2.3, some notations should be clarified in the context IVHF information systems at first. In specific, we let $\mathbb{E}$ and $\mathbb{E}^c$ be a M&A target that belongs to and not belongs to $\mathbb{E}$ respectively. Next, the conditional probability of a M&A target x to a standard set $\mathbb{E}$ is denoted by $\Pr(\mathbb{E}|x)$, whereas $\Pr(\mathbb{E}^c|x)$ denotes the conditional probability of x to $\mathbb{E}^c$. In light of the above-stated notations, we present the following estimated costs of choosing diverse actions for x:

$$\mathcal{L}(a_P|x) = \lambda_{PP}Pr(\mathbb{E}|x) + \lambda_{PN}Pr(\mathbb{E}^c|x); \tag{31}$$

$$\mathcal{L}(a_B|x) = \lambda_{BP} Pr(\mathbb{E}|x) + \lambda_{BN} Pr(\mathbb{E}^c|x); \tag{32}$$

$$\mathcal{L}(a_N|x) = \lambda_{NP} Pr(\mathbb{E}|x) + \lambda_{NN} Pr(\mathbb{E}^c|x). \tag{33}$$

Then, the following minimum-risk decision rules can be obtained according to Bayesian decision procedures. (P_2) If $x \in POS(G)$ is reached such that $\mathcal{L}(a_P|x) < \mathcal{L}(a_B|x)$ and $\mathcal{L}(a_P|x) < \mathcal{L}(a_N|x)$ hold.

(B_2) If $x \in BND(G)$ is reached such that $\mathcal{L}(a_B|x) < \mathcal{L}(a_P|x)$ and $\mathcal{L}(a_B|x) < \mathcal{L}(a_N|x)$ hold.

(N_2) If $x \in NEG(G)$ is reached such that $\mathcal{L}(a_N|x) < \mathcal{L}(a_P|x)$ and $\mathcal{L}(a_N|x) < \mathcal{L}(a_B|x)$ hold.

Compared with classical DTRSs, since the equation $\Pr(\mathbb{E}|x) + \Pr(\mathbb{E}^c|x) = \{[1,1]\}$ does not hold in general. Hence, considering that IVHFSs is an extended form of IVF sets (IVFSs), we utilize the updated 3WD strategy of DTRSs in the IVF context [10,18,49,50]. Moreover, we also assume $\lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$, $\lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$, $(\lambda_{PN} - \lambda_{BN})(\lambda_{NP} - \lambda_{BP}) \ge (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$ and $\alpha > \beta$ usually hold in plenty of realistic decision making situations. Then, a series of relatively simplified decision rules in the context of IVHFSs can be further obtained with $\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{\lambda_{BP} - \lambda_{PP}}$ and $\beta = \frac{\lambda_{BN} - \lambda_{NN}}{\lambda_{NP} - \lambda_{BP}}$. (P_3) If $x \in POS(G)$ is reached such that $\frac{\Pr(\mathbb{E}|x)}{\Pr(\mathbb{E}^c|x)} > \alpha$ holds.

(B_3) If $x \in BND(G)$ is reached such that $\beta < \frac{\Pr(\mathbb{E}|x)}{\Pr(\mathbb{E}^c|x)} < \alpha$ holds.

(N_3) If $x \in NEG(G)$ is reached such that $\frac{\Pr(\mathbb{E}|x)}{\Pr(\mathbb{E}^c|x)} \le \beta$ holds.

Remark 4.1. In light of four different kinds of theoretical models, the conditional probability $\Pr(\mathbb{E}|x)$ and $\Pr(\mathbb{E}^c|x)$ can be substituted by corresponding multiple IVHF membership degrees presented from Section 3.1 to Section 3.4. For instance, by virtue of type-I IVHF MG-DTRSs over two universes, $\Pr(\mathbb{E}|x)$ can be substituted by $\Omega_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x)$ or $\Psi_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x)$, whereas $\Pr(\mathbb{E}^c|x)$ can be substituted by $\Omega_{\mathbb{E}^c}^{\sum_{i=1}^m \mathbb{R}_i}(x)$ or $\Psi_{\mathbb{E}^c}^{\sum_{i=1}^m \mathbb{R}_i}(x)$; by virtue of type-II IVHF MG-DTRSs over two universes, $\Pr(\mathbb{E}|x)$ can be substituted by $\Xi_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x)$, whereas $\Pr(\mathbb{E}^c|x)$ can be substituted by $\Xi_{\mathbb{E}^c}^{\sum_{i=1}^m \mathbb{R}_i}(x)$; by virtue of type-III IVHF MG-DTRSs over two universes, $\Pr(\mathbb{E}|x)$ can be substituted by $Z_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x)$, whereas $\Pr(\mathbb{E}^c|x)$ can be substituted by $Z_{\mathbb{E}^c}^{\sum_{i=1}^m \mathbb{R}_i}(x)$; by virtue of type-IV IVHF MG-DTRSs over two universes, $\Pr(\mathbb{E}|x)$ can be substituted by $\Phi_{\mathbb{E}}^{\sum_{i=1}^m \mathbb{R}_i}(x)$, whereas $\Pr(\mathbb{E}^c|x)$ can be substituted by $\Phi_{\mathbb{E}^c}^{\sum_{i=1}^m \mathbb{R}_i}(x)$. After the substitution of $\Pr(\mathbb{E}|x)$ and $\Pr(\mathbb{E}^c|x)$, we can compute the score values of $\frac{\Pr(\mathbb{E}|x_j)}{\Pr(\mathbb{E}^c|x_j)}$ $(j = 1, 2, \ldots, p)$ by Definition 2.3, and further obtain the priority of optional M&A targets x_j by ranking $s(\frac{\Pr(\mathbb{E}|x_j)}{\Pr(\mathbb{E}^c|x_j)})$, then the optimal M&A target x^* can be determined as $x^* = \max_{j=1}^{p}\{s(\frac{\Pr(\mathbb{E}|x_j)}{\Pr(\mathbb{E}^c|x_j)})\}$.

Remark 4.2. In practical M&A target selection situations, according to general risk appetites, the board of directors of acquiring companies needs to choose a information fusion tactic from four kinds of theoretical models. Moreover, it is noted that type-I IVHF MG-DTRSs over two universes include the optimistic version and the pessimistic version, which indicate a totally optimistic information fusion tactic and a totally pessimistic information fusion tactic respectively. Type-II IVHF MG-DTRSs over two universes use an information fusion tactic that tends to select groups' primary opinions. Type-III IVHF MG-DTRSs over two universes use an information fusion tactic that tends to select personal primary opinions. Type-IV IVHF MG-DTRSs over two universes act as a compromise version that takes advantages of type-II and type-III presented models.

Remark 4.3. In general, this work first develops four IVHF MG-DTRSs over two universes models based on different strategies when handling consensus and conflicting opinions provided by different decision makers in a typical group decision making process. Afterwards, the corresponding MAGDM methods are further proposed by virtue of the developed theoretical models. At last, the effectiveness of the proposed MAGDM methods is illustrated in the M&A target selections background. It is worth pointing out other than the M&A target selections background, the proposed MAGDM methods are also suitable for addressing other typical MAGDM situations if we can transform the situations to the information system described in this section, i.e., the proposed MAGDM methods act as a general strategy that can be applied to a series of MAGDM situations. In addition, in order to solve the challenge of incomplete, imprecise and hesitant expression of attributes, the developed theoretical models are studied in the IVHF context. It is noted that the validity and effectiveness of the proposed methods also hold if we select other generalized fuzzy decision making information. The compatibility of the proposed methods lies in the core steps of M&A target selections remain unchanged, and the only thing that gets changed is the operational laws of the generalized fuzzy sets.

4.3. Algorithm for M&A target selections

In what follows, we summarize a general algorithm for addressing the situation of M&A target selections.

Input An information system of M&A target selections $(U, V, \mathbb{R}_i, \mathbb{E})$.

Output The optimal M&A target.

Step 1 Choose a suitable information fusion tactic according to general risk appetites of the board of directors.

Step 2 Determine the weight value of each decision maker and the parameter λ if the information fusion tactic is chosen from type-II, type-III and type-IV IVHF MG-DTRSs over two universes.

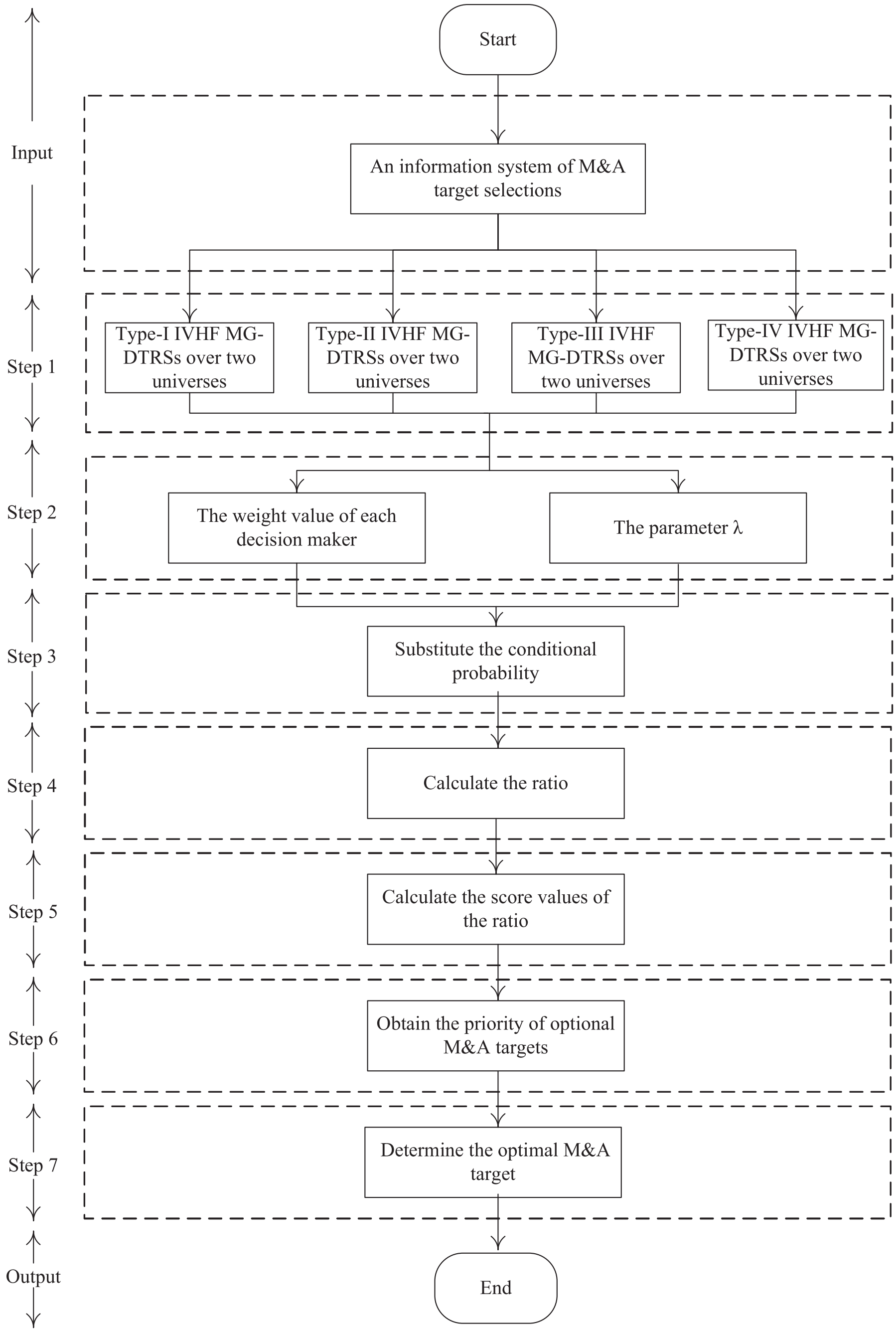

Fig. 1. The summary of the presented algorithm for M&A target selections.

Step 3 Substitute the conditional probability $\Pr(\mathbb{E}|x)$ and $\Pr(\mathbb{E}^c|x)$ by the chosen multiple IVHF membership degrees.
Step 4 Calculate the ratio $\frac{\Pr(\mathbb{E}|x_j)}{\Pr(\mathbb{E}^c|x_j)}$ for each optional M&A target x_j.
Step 5 Calculate the score values of $\frac{\Pr(\mathbb{E}|x_j)}{\Pr(\mathbb{E}^c|x_j)}$.
Step 6 Obtain the priority of optional M&A targets x_j by ranking $s(\frac{\Pr(\mathbb{E}|x_j)}{\Pr(\mathbb{E}^c|x_j)})$.
Step 7 Determine the optimal M&A target x^* according to $x^* = \max_{j=1}^{p}\{s(\frac{\Pr(\mathbb{E}|x_j)}{\Pr(\mathbb{E}^c|x_j)})\}$.
The flow chart of the proposed algorithm for M&A target selections is summarized in Fig. 1.

Table 1
IVHF relationship between optional M&A targets and assessment factors given by the first expert.

$\mathbb{R}_1$	y_1	y_2	y_3	y_4	y_5
x_1	{[0.7, 0.8], [0.8, 0.9]}	{[0.2, 0.3], [0.3, 0.4]}	{[0.6, 0.7], [0.6, 0.8]}	{[0.5, 0.6], [0.5, 0.7]}	{[0.2, 0.3], [0.3, 0.4]}
x_2	{[0.4, 0.5], [0.5, 0.6]}	{[0.3, 0.4], [0.3, 0.5]}	{[0.5, 0.6], [0.6, 0.7]}	{[0.5, 0.6], [0.6, 0.7]}	{[0.6, 0.7], [0.7, 0.8]}
x_3	{[0.6, 0.7], [0.7, 0.8]}	{[0.7, 0.8], [0.7, 0.9]}	{[0.3, 0.4], [0.3, 0.5]}	{[0.2, 0.3], [0.3, 0.4]}	{[0.5, 0.6], [0.6, 0.7]}
x_4	{[0.7, 0.9], [0.8, 0.9]}	{[0.4, 0.5], [0.5, 0.6]}	{[0.3, 0.4], [0.4, 0.5]}	{[0.5, 0.6], [0.6, 0.7]}	{[0.6, 0.7], [0.6, 0.8]}
x_5	{[0.7, 0.8], [0.8, 0.9]}	{[0.3, 0.4], [0.4, 0.5]}	{[0.5, 0.6], [0.6, 0.7]}	{[0.6, 0.7], [0.7, 0.8]}	{[0.7, 0.8], [0.8, 0.9]}

Table 2
IVHF relationship between optional M&A targets and assessment factors given by the second expert.

$\mathbb{R}_2$	y_1	y_2	y_3	y_4	y_5
x_1	{[0.6, 0.7], [0.7, 0.8]}	{[0.2, 0.4], [0.3, 0.4]}	{[0.6, 0.7], [0.7, 0.8]}	{[0.5, 0.6], [0.6, 0.7]}	{[0.2, 0.4], [0.3, 0.4]}
x_2	{[0.4, 0.5], [0.4, 0.6]}	{[0.3, 0.4], [0.4, 0.5]}	{[0.5, 0.6], [0.5, 0.7]}	{[0.5, 0.7], [0.6, 0.7]}	{[0.7, 0.8], [0.8, 0.9]}
x_3	{[0.6, 0.7], [0.6, 0.8]}	{[0.7, 0.9], [0.8, 0.9]}	{[0.3, 0.4], [0.4, 0.5]}	{[0.2, 0.4], [0.3, 0.4]}	{[0.4, 0.5], [0.5, 0.6]}
x_4	{[0.7, 0.8], [0.7, 0.9]}	{[0.5, 0.6], [0.6, 0.7]}	{[0.3, 0.5], [0.4, 0.5]}	{[0.5, 0.7], [0.6, 0.7]}	{[0.6, 0.7], [0.7, 0.8]}
x_5	{[0.7, 0.8], [0.8, 0.9]}	{[0.4, 0.5], [0.5, 0.6]}	{[0.5, 0.7], [0.6, 0.7]}	{[0.6, 0.8], [0.7, 0.8]}	{[0.7, 0.8], [0.7, 0.9]}

Table 3
IVHF relationship between optional M&A targets and assessment factors given by the third expert.

$\mathbb{R}_3$	y_1	y_2	y_3	y_4	y_5
x_1	{[0.6, 0.7], [0.6, 0.8]}	{[0.3, 0.4], [0.3, 0.5]}	{[0.6, 0.8], [0.7, 0.8]}	{[0.6, 0.7], [0.7, 0.8]}	{[0.3, 0.4], [0.3, 0.5]}
x_2	{[0.4, 0.5], [0.5, 0.6]}	{[0.3, 0.5], [0.4, 0.6]}	{[0.5, 0.6], [0.6, 0.7]}	{[0.5, 0.6], [0.6, 0.7]}	{[0.7, 0.8], [0.7, 0.9]}
x_3	{[0.7, 0.8], [0.7, 0.9]}	{[0.7, 0.8], [0.8, 0.9]}	{[0.3, 0.5], [0.4, 0.5]}	{[0.3, 0.4], [0.3, 0.5]}	{[0.3, 0.4], [0.4, 0.5]}
x_4	{[0.6, 0.8], [0.7, 0.8]}	{[0.5, 0.6], [0.5, 0.7]}	{[0.4, 0.5], [0.4, 0.6]}	{[0.5, 0.6], [0.6, 0.7]}	{[0.5, 0.7], [0.6, 0.7]}
x_5	{[0.7, 0.8], [0.7, 0.9]}	{[0.5, 0.6], [0.5, 0.7]}	{[0.5, 0.6], [0.5, 0.7]}	{[0.5, 0.7], [0.6, 0.7]}	{[0.7, 0.8], [0.8, 0.9]}

5. An illustrative case study

In the following section, we plan to show the validity of the presented algorithm for a M&A target selections problem (adapted from Literature [44]) presented in the previous section. Moreover, a sensitivity analysis together with a comparative analysis is scheduled to illustrate the effectiveness of the established MAGDM approaches.

5.1. Case description

Suppose a coal corporate acquirer plans to determine optimal M&A acquiring firms. For the sake of assuring the justice of the M&A process, the acquiring firm invites three experts to provide the information system of M&A target selections $(U, V, \mathbb{R}_i, \mathbb{E})$. In specific, $U = \{x_1, x_2, x_3, x_4, x_5\}$ denotes the set of optional M&A targets, and $V = \{y_1(mineral\ output), y_2(mining\ difficulty), y_3(proved\ reserves), y_4(reserve-production\ ratio), y_5(science\ and\ technology\ contribution\ rate)\}$ denotes the set of assessment factors. According to universes U and V, the relationship between optional M&A targets and assessment factors $\mathbb{R}_i \in IVHFR(U \times V)$ can be established by each invited expert, which is expressed in Tables 1–3 as follows. In addition, the same board of directors of the coal corporate acquirer can also confirm a standard set $\mathbb{E} = \{\langle y_1, \{[0.7, 0.8]\}\rangle, \langle y_2, \{[0.5, 0.6]\}\rangle, \langle y_3, \{[0.7, 0.8]\}\rangle, \langle y_4, \{[0.2,\ 0.3]\}\rangle, \langle y_5, \{[0.4,\ 0.5]\}\rangle\}$. In light of the above statements, the information system of M&A target selections $(U, V, \mathbb{R}_i, \mathbb{E})$ $(i = 1, 2, 3)$ can be constructed.

5.2. M&A target selection procedures

In the following, we use the algorithm for M&A target selections to address the case study presented in Section 5.1.

First, suppose the board of directors intends to take advantages of both groups' primary opinions and personal primary opinions, type-IV IVHF MG-DTRSs over two universes can be chosen as a suitable information fusion tactic.

Then, we let the weight value of each decision maker is equal, i.e., $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$. In addition, we suppose the parameter $\lambda = 1$.

Next, the conditional probability $\Pr(\mathbb{E}|x)$ and $\Pr(\mathbb{E}^c|x)$ are substituted by $\Phi_{\mathbb{E}}^{\sum_{i=1}^{m}\mathbb{R}_i}(x)$ and $\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{m}\mathbb{R}_i}(x)$ as follows:

$$\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_1) = GIVHFHAG_1\left(\Theta_{\mathbb{E}}^{\mathbb{R}_1}(x_1), \Theta_{\mathbb{E}}^{\mathbb{R}_2}(x_1), \Theta_{\mathbb{E}}^{\mathbb{R}_3}(x_1)\right)$$
$$= \left\{\left[\left(1 - \prod_{i=1}^{3}\left(1 - \left(\Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}L}(x)\right)\right)^{1/3}\right), \left(1 - \prod_{i=1}^{3}\left(1 - \left(\Theta_{\mathbb{E}}^{\mathbb{R}_{\tau(i)}U}(x)\right)\right)^{1/3}\right)\right]\right\}$$
$$= \{[0.8015, 0.9220], [0.8376, 0.9487]\}.$$

Table 4
Comparison of different theoretical models.

Different kinds of IVHF MG-DTRSs over two universes	Ranking result of optional M&A targets
Type-I IVHF MG-DTRSs over two universes (the optimistic version)	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$
Type-I IVHF MG-DTRSs over two universes (the pessimistic version)	$x_1 \succ x_3 \succ x_4 \succ x_5 \succ x_2$
Type-II IVHF MG-DTRSs over two universes	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$
Type-III IVHF MG-DTRSs over two universes	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$
Type-IV IVHF MG-DTRSs over two universes	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$

Similarly, $\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_2) = \{[0.7641, 0.8934], [0.8064, 0.9325]\}$, $\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_3) = \{[0.7994, 0.9174], [0.8285, 0.9460]\}$, $\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_4) = \{[0.8029, 0.9319], [0.8406, 0.9504]\}$, $\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_5) = \{[0.8406, 0.9438], [0.8750, 0.9670]\}$, $\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_1) = \{[0.6392, 0.8088], [0.6902, 0.8657]\}$, $\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_2) = \{[0.7090, 0.8630], [0.7597, 0.9017]\}$, $\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_3) = \{[0.6270, 0.8075], [0.6819, 0.8498]\}$, $\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_4) = \{[0.7118, 0.8608], [0.7635, 0.9015]\}$, $\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_5) = \{[0.7542, 0.9027], [0.8029, 0.9288]\}$.

On the basis of the above results, we calculate the ratio $\frac{\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x)}{\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x)}$ for each optional M&A target x_j as follows:

$$\frac{\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_1)}{\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_1)} = \{[1.0959, 1.2136], [1.1399, 1.2538]\},$$

$$\frac{\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_2)}{\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_2)} = \{[1.0341, 1.0614], [1.0352, 1.0777]\},$$

$$\frac{\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_3)}{\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_3)} = \{[1.1132, 1.2150], [1.1362, 1.2751]\},$$

$$\frac{\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_4)}{\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_4)} = \{[1.0543, 1.1009], [1.0825, 1.1280]\},$$

$$\frac{\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_5)}{\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x_5)} = \{[1.0410, 1.0898], [1.0455, 1.1145]\}.$$

At last, we obtain the priority of optional M&A targets x_j by ranking $s(\frac{\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x)}{\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x)})$, and determine the optimal M&A target x^* according to $x^* = \max_{j=1}^{p}\{s(\frac{\Phi_{\mathbb{E}}^{\sum_{i=1}^{3}\mathbb{R}_i}(x)}{\Phi_{\mathbb{E}^c}^{\sum_{i=1}^{3}\mathbb{R}_i}(x)})\}$. We can find that the ranking result is $x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$, and the optimal M&A target is x_3.

5.3. Sensitivity analysis

In what follows, we plan to conduct a sensitivity analysis to explain the influence of utilizing different theoretical models and the parameter λ. On one hand, by choosing the suitable information fusion tactic as type-I, type-II and type-III theoretical models respectively, we use the algorithm for M&A target selections to address the identical case study if $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$ and $\lambda = 1$ hold, and the ranking result of optional M&A targets is presented in Table 4.

We can find that the ranking conclusion of optional M&A targets by using optimistic theoretical models, type-II theoretical models, type-III theoretical models and type-IV theoretical models is identical. In addition, the ranking result of optional M&A targets by virtue of pessimistic theoretical models is slightly different from the previous result, and the optimal M&A target is x_1. The cause of this inconsistent ranking result is that the absoluteness and extremeness of the pessimistic information fusion tactic, which affect the final decision conclusion.

Table 5
Comparison of different λ by using Type-IV IVHF MG-DTRSs over two universes.

Different λ by using Type-IV IVHF MG-DTRSs over two universes	Ranking result of optional M&A targets
$\lambda = 1$	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$
$\lambda = 2$	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$
$\lambda = 5$	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$
$\lambda = 10$	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$
$\lambda = 20$	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$
$\lambda = 50$	$x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_2$

On the other hand, by changing the parameter λ, we use the algorithm for M&A target selections to address the identical case study by virtue of type-IV IVHF MG-DTRSs over two universes if $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$ holds, and the ranking conclusion of optional M&A targets is presented in Table 5.

We can find that the ranking conclusion of optional M&A targets is not sensitive to diverse values of the parameter λ, i.e., the optimal M&A target remains identical when the parameter λ changes. Thus, the conclusion of M&A target selections is reliable in this case study regardless of the change of the parameter λ.

5.4. Comparison analysis and discussions

In this section, we plan to conduct a comparison analysis along with related discussions to show the validity of the above-presented conclusions.

5.4.1. Comparison analysis with the MAGDM method presented in literature [44]

Taking advantages of multi-granularity (MG) computing, a MAGDM approach by using MGRSs over two universes in the Pythagorean fuzzy (PF) background is explored. In the following, the approach proposed in Literature [44] is used to address the M&A target selection problem presented in Section 5.1.

First, two approximations of optimistic IVHF MGRSs over two universes for the a standard set $\mathbb{E}$ can be computed.

$$\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E}) = \{\langle x_1, \{[0.3, 0.5], [0.4, 0.5]\}\rangle, \langle x_2, \{[0.3, 0.4], [0.4, 0.5]\}\rangle, \langle x_3, \{[0.5, 0.6]\}\rangle, \langle x_4, \{[0.3, 0.4], [0.4, 0.5]\}\rangle, \langle x_5, \{[0.3, 0.4], [0.3, 0.5]\}\rangle\},$$

$$\overline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E}) = \{\langle x_1, \{[0.2, 0.3], [0.3, 0.4]\}\rangle, \langle x_2, \{[0.3, 0.4], [0.3, 0.5]\}\rangle, \langle x_3, \{[0.4, 0.5]\}\rangle, \langle x_4, \{[0.3, 0.4], [0.3, 0.5]\}\rangle, \langle x_5, \{[0.2, 0.3], [0.2, 0.4]\}\rangle\}.$$

Similarly, two approximations of pessimistic IVHF MGRSs over two universes for the a standard set $\mathbb{E}$ can be computed as well.

$$\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E}) = \{\langle x_1, \{[0.6, 0.7], [0.7, 0.8]\}\rangle, \langle x_2, \{[0.5, 0.6], [0.5, 0.7]\}\rangle, \langle x_3, \{[0.6, 0.7], [0.7, 0.8]\}\rangle, \langle x_4, \{[0.6, 0.8], [0.7, 0.8]\}\rangle, \langle x_5, \{[0.7, 0.8]\}\rangle\},$$

$$\overline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E}) = \{\langle x_1, \{[0.7, 0.8]\}\rangle, \langle x_2, \{[0.5, 0.6], [0.6, 0.7]\}\rangle, \langle x_3, \{\{[0.7, 0.8]\}\}\rangle, \langle x_4, \{\{[0.7, 0.8]\}\}\rangle, \langle x_5, \{[0.7, 0.8]\}\rangle\}.$$

Then, we compute the compromise evaluation set $(\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E})) \boxplus (\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E}))$ by using four approximations $\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E})$, $\overline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E})$, $\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E})$ and $\overline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E})$.

$$\left(\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E})\right) \boxplus \left(\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E})\right)$$
$$= \{\langle x_1, \{[0.9328, 0.9790], [0.9622, 0.9880]\}\rangle, \langle x_2, \{[0.8775, 0.9424], [0.9160, 0.9775]\}\rangle, \langle x_3, \{\{[0.9640, 0.9880], [0.9730, 0.9920]\}\}\rangle, \langle x_4, \{[0.9412, 0.9856], [0.9622, 0.9900]\}\rangle, \langle x_5, \{[0.9496, 0.9832], [0.9496, 0.9880]\}\rangle\}.$$

At last, in light of the above results, we further rank optional M&A targets according to the score function of $(\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3}\mathbb{R}_i}^{O}(\mathbb{E})) \boxplus (\underline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3}\mathbb{R}_i}^{P}(\mathbb{E}))$, then we can find that the ranking result is $x_3 \succ x_4 \succ x_1 \succ x_5 \succ x_2$, and the optimal M&A target is x_3.

5.4.2. Comparison analysis with the MAGDM method presented in literature [42]

Considering divergences of opinions between different experts, Literature [42] proposed a MAGDM method by means of index sets. In the following, the approach proposed in Literature [42] is used to address the M&A target selection problem presented in Section 5.1.

First, we compute the index sets of $\underline{\sum_{i=1}^{3} \mathbb{R}_i}^{O}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3} \mathbb{R}_i}^{O}(\mathbb{E})$, $\underline{\sum_{i=1}^{3} \mathbb{R}_i}^{P}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3} \mathbb{R}_i}^{P}(\mathbb{E})$ and $(\underline{\sum_{i=1}^{3} \mathbb{R}_i}^{O}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3} \mathbb{R}_i}^{O}(\mathbb{E})) \boxplus (\underline{\sum_{i=1}^{3} \mathbb{R}_i}^{P}(\mathbb{E}) \boxplus \overline{\sum_{i=1}^{3} \mathbb{R}_i}^{P}(\mathbb{E}))$.

$$\mathcal{T}_1 = \left\{ j | \max_{x_j \in U} \left\{ \underline{\sum_{i=1}^{3} \mathbb{R}_i}^{O}(\mathbb{E})(x_j) \boxplus \overline{\sum_{i=1}^{3} \mathbb{R}_i}^{O}(\mathbb{E})(x_j) \right\} \right\} = \{3\},$$

$$\mathcal{T}_2 = \left\{ j' | \max_{x_{j'} \in U} \left\{ \underline{\sum_{i=1}^{3} \mathbb{R}_i}^{P}(\mathbb{E})(x_{j'}) \boxplus \overline{\sum_{i=1}^{3} \mathbb{R}_i}^{P}(\mathbb{E})(x_{j'}) \right\} \right\} = \{5\},$$

$$\mathcal{T}_3 = \left\{ j'' | \max_{x_{j''} \in U} \left\{ \left(\underline{\sum_{i=1}^{3} \mathbb{R}_i}^{O}(\mathbb{E})(x_{j''}) \boxplus \overline{\sum_{i=1}^{3} \mathbb{R}_i}^{O}(\mathbb{E})(x_{j''}) \right) \boxplus \left(\underline{\sum_{i=1}^{3} \mathbb{R}_i}^{P}(\mathbb{E})(x_{j''}) \boxplus \overline{\sum_{i=1}^{3} \mathbb{R}_i}^{P}(\mathbb{E})(x_{j''}) \right) \right\} \right\} = \{3\}.$$

Then, according to the decision rules that originate from risk decision making strategies in the discipline of operational research, we can obtain that $\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathcal{T}_3 = \emptyset$ and $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ hold, which indicate there exist substantial divergences of opinions between different experts. Finally, we utilize $\mathcal{T}_3 = \{3\}$ as the guideline for M&A target selections, i.e., the enterprise x_3 can be regarded as the optimal M&A target.

5.4.3. Discussions

Based on the decision results for M&A target selections by using MAGDM approaches presented in Literature [44] and Literature [42], both these two decision results indicate the optimal M&A target is x_3, which is identical with the decision result by means of type-IV theoretical models. In addition, there exists a slight difference between the decision result by using Literature [44] and the decision result by means of type-IV theoretical models, i.e., the previous one shows x_4 is superior to x_1, whereas the latter one indicates x_4 is inferior to x_1. Compared with Literature [44] and Literature [42], the reason for this difference lies in the MAGDM approach by means of type-IV IVHF MG-DTRSs over two universes owns effective data-driven error-tolerance scheme and considers risk preferences of different experts. In specific, we sum up the following merits of the established M&A target selection rule:

(1) The proposed M&A target selection methods take merits of IVHFSs, MGRSs over two universes and DTRSs at the same time, which can not only depict incomplete, imprecise and hesitant information existed in various real-world procedures of M&A target selections, but also provide a viable and effective MAGDM guideline by virtue of MG-3WD frameworks.
(2) The proposed M&A target selection methods offer four different theoretical MG-3WD models according to different risk preferences of decision makers, which include absolute optimistic and pessimistic versions, groups' primary opinions-oriented versions, personal primary opinions-oriented versions, and relative compromise versions.
(3) The proposed M&A target selection methods increase the performance of dealing with noisy data by introducing minimum-risk decision rules that obtained from Bayesian decision rules. Moreover, the case study shows that the constructed MAGDM methods by using the presented theoretical models can substantially increase the efficiency and reduce the uncertainty of M&A target selections.

6. Conclusions

In the present work, for the sake of addressing challenges of consensus processes in complicated MAGDM situations, a series of novel theoretical models titled IVHF MG-DTRSs over two universes are detailedly studied in accordance with MG-3WD frameworks. In specific, not only fundamental definitions, but also several significant propositions of four kinds of theoretical models are studied. Moreover, in light of the proposed theoretical models, we investigate general MAGDM approaches for M&A target selections. At last, a realistic case study, a sensitivity analysis and a comparative analysis are conducted in the background of M&A target selections to testify the correctness and validity of the established MAGDM approaches.

With regard to the future study, we may need to resolve the following challenges: (1) it is desirable to further discuss other theoretical aspects such as uncertainty measures, granular structures, attribute reductions, clustering algorithms, and so forth; (2) identifying practical decision making scenarios in which IVHF information systems produce uncertainties and applying them to IVHF models are a promising study topic; (3) we aim to explore viable schemes to prove that results outperform the ones obtained with other decision making approaches; (4) integrating hybrid intelligent methods such as graph theory and soft sets to cope with a greater range of complicated MAGDM problems is also worth looking into; (5) new decision making algorithms should be more creative than simple and direct extensions of existing ones by using MG-3WD frameworks.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this work.

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