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Formal concept analysis based on fuzzy granularity base for different granulations

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Abstract

This paper introduces granular computing (GrC) into formal concept analysis (FCA). It provides a unified model for concept lattice building and rule extraction on a fuzzy granularity base for different granulations. One of the strengths of GrC is that larger granulations help to hide some specific details, whereas FCA in a GrC context can prevent losses due to concept lattice complexity. However, the number of superfluous rules increases exponentially with the scale of the decision context. To overcome this we present some inference rules and maximal rules and prove that the set of all these maximal rules is complete and nonredundant. Thus, users who want to obtain decision rules should generate maximal rules. Examples demonstrate that application of the method is valid and practicable. In summary, this approach utilizes FCA in a GrC context and provides a practical basis for data analysis and processing.

Keywords: Formal concept analysis; Granular computing; Fuzzy equivalence relation; Rule extraction; Decision inference

1. Introduction

Formal concept analysis (FCA) is an order-theoretic method for mathematical analysis of scientific data that was pioneered by Wille in 1982 [28]. FCA comes from a philosophical understanding and a concept, viewed as the formalization of a unit of thought, consists of extent and intent. The extent is understood as the collection of all objects belonging to the concept, while the intent is the multitude of all attributes common to all those objects. A concept lattice is an effective tool in FCA and is very suitable for mining potential concepts of data sets. In recent years, FCA has been extensively studied [2–4,9–11,16,19,25] and has become a powerful tool for machine learning, software engineering and information retrieval.

Although FCA with its strong mathematical foundation can easily generate induced concepts from a given data set, a great number of concepts will be produced when the quantity of objects or attributes is very large, and useful concepts may be overwhelmed. To avoid this problem, granular computing (GrC), described by Zadeh in 1996 [31], is introduced into FCA in this paper. This addresses the impacts of complex data sets to some extent in that larger granulations can help to hide some specific details and the problem can be solved from the overall picture. As an

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Table 1
A sample data set.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	×	×			×	×	×	×
2			×					×
3	×		×			×	×	
4		×						
5		×	×	×	×	×	×	×
6			×		×			
7		×	×	×		×		×
8	×	×		×	×	×	×	

effective tool with vast potential for knowledge acquisition, GrC has been widely investigated by researchers in the field of artificial intelligence [5,15,17,18,20,21,24].

In addition, the fuzzy equivalence relation [30] of fuzzy set theory is introduced into FCA, and we propose a model of FCA based on the fuzzy equivalence for different granulations. Introduction of fuzzy equivalence relations into FCA was inspired by several studies. Aswani Kumar et al. proposed a method based on fuzzy *k*-means clustering to reduce the size of the concept lattice, and applied the proposed method to information retrieval and information visualization [1]. Three types of variable threshold concept lattice were defined, with properties analogous to those of classical concept lattices [32]. Wu et al. presented a general framework for the study of fuzzy rough sets in which both constructive and axiomatic approaches were used [29]. Fan et al. proposed a new form of fuzzy concept lattice and used three types of known fuzzy concept lattice and the new fuzzy concept lattice to propose two coherent fuzzy inference methods (lower and upper approximate fuzzy inferences) [7]. Based on both lattice-theoretic and fuzzy set-theoretic operators, two new pairs of rough fuzzy set approximations within a fuzzy formal context were defined [23]. Li and Zhang proposed a type of fuzzy concept lattice by T implication in a fuzzy formal context at the level $\delta \in I_T$, and investigated δ-reducts [13].

Normally, existing knowledge acquisition models only adopt different logic methods locally and cannot provide a uniform technology for knowledge acquisition overall. To avoid the problem mentioned above, a unified method based on fuzzy granularity base for different granulations is proposed. The fuzzy granularity base not only offers a uniform technology for knowledge acquisition overall, but also facilitates knowledge sharing.

The remainder of the paper is organized as follows. Section 2 describes basic FCA notions. Section 3 constructs a fuzzy granularity base with different granulations. Section 4 builds a δ-concept lattice based on fuzzy granularity with granulation $\rho(R_\delta)$, and extracts a complete and nonredundant set of δ-rules from the δ-concept. Section 5 extracts (θ, σ)-rules from a (θ, σ)-decision context based on fuzzy granularity bases with multi-granulations, which yields a τ-complete and τ-nonredundant set of (θ, σ)-rules. Section 6 discusses perspectives for further work.

2. Basic notions of FCA

This section only provides the most basic FCA notions. For more extensive information, readers are referred to Ganter and Wille [8].

A formal context is a triplet $K = (G, M, I)$, where G and M are sets, and $I \subseteq G \times M$ is a binary relation. In this case, members of G are called objects and members of M are called attributes, and I is viewed as an incidence relation between objects and attributes. Accordingly, gIm or $(g, m) \in I$ denotes "object g has attribute m".

Formal contexts are mostly represented by rectangular tables in which row headers are object names and column headers are attribute names. In a table, a cross denotes that the row object has the column attribute. For example, a formal context for a grocery shop is shown in Table 1. The set of objects G is composed of $1, 2, \ldots, 8$ denoting customer codes, while the set M of attributes comprises $a_1, a_2, \ldots, a_8$ to denote commodity codes. If customer i purchases commodity a_j, this is denoted as $(i, a_j) \in I$, which is shown in the table by "×".

In a formal context $K = (G, M, I)$, for a set $A \subseteq G$ we define

$$A^* = \{m \in M | gIm, \forall g \in A\}.$$

Correspondingly, for a set $B \subseteq M$ we define

$$B^* = \{g \in G | gIm, \forall m \in B\}.$$

(A, B) is called a formal concept if $A^* = B$ and $B^* = A$. In this case, A is called the extent and B is called the intent of (A, B). The set of all concepts is denoted as $\mathscr{B}(K)$. If $(A_1, B_1), (A_2, B_2) \in \mathscr{B}(K)$, we define

$$(A_1, B_1) \preceq (A_2, B_2) \;\Leftrightarrow\; A_1 \subseteq A_2 \;\Leftrightarrow\; B_1 \supseteq B_2.$$

In this case, (A_2, B_2) is a superconcept of (A_1, B_1), (A_1, B_1) is a subconcept of (A_2, B_2), and "$\preceq$" is a hierarchical order of concepts. The set of all concepts ordered in this way is called the concept lattice induced from K, which is still denoted as $\mathscr{B}(K)$ in the absence of ambiguity. In addition, let $B_1, B_2 \subseteq M$. If $B_1^* \subseteq B_2^*$, then we say $B_1 \to B_2$ is a rule of K.

Proposition 1. *In a formal context (G, M, I), let $A, A_1, A_2 \subseteq G$ be sets of objects and let $B, B_1, B_2 \subseteq M$ be sets of attributes. Then*

$$\begin{array}{ll} (1)\ A_1 \subseteq A_2 \Rightarrow A_2^* \subseteq A_1^* & (2)\ B_1 \subseteq B_2 \Rightarrow B_2^* \subseteq B_1^*, \\ (3)\ A \subseteq A^{**};\ B \subseteq B^{**} & (4)\ A^* = A^{***};\ B^* = B^{***}. \end{array}$$

3. Fuzzy granularity base for different granulations

Let X and Y be two finite and nonempty sets. A fuzzy subset $\tilde{R} \in \mathscr{F}(X \times Y)$ is referred to as a fuzzy relation from X to Y. $\tilde{R}(x, y)$ is the degree of relation between x and y, where $(x, y) \in X \times Y$. If $X = Y$, then $\tilde{R}$ is referred to as a fuzzy relation on X. Let $\tilde{R}$ be a fuzzy relation on X. $\tilde{R}$ is a reflexive fuzzy relation if $\tilde{R}(x, x) = 1$ for all $x \in X$. $\tilde{R}$ is a symmetric fuzzy relation if $\tilde{R}(x, y) = \tilde{R}(y, x)$ for all $x, y \in X$. $\tilde{R}$ is a transitive fuzzy relation if $\tilde{R}(x, z) \geq \bigvee_{y \in X}(\tilde{R}(x, y) \wedge \tilde{R}(y, z))$ for all $x, z \in X$. $\tilde{R}$ is a fuzzy equivalence relation if $\tilde{R}$ is a reflexive, symmetric and transitive fuzzy relation. It is easy to see that $\tilde{R}$ is a fuzzy equivalence relation iff $\tilde{R}_a$ is an equivalence ordinary binary relation for all $a \in [0, 1]$.

Fuzzy granularity base: In a formal context $K = (G, M, I)$, if R is a fuzzy equivalence relation on M, then $\Sigma = (M, R)$ is referred to as a fuzzy granularity base, where R is obtained by the method below.

The similarity between any two elements $a_i, a_j \in M$ is defined as

$$r_{ij} = \frac{|a_i^* \cap a_j^*|}{|a_i^* \cup a_j^*|}.$$

Obviously, the fuzzy relation (or matrix)

$$\tilde{R} = (r_{ij})_{|M| \times |M|}$$

is a reflexive and symmetric fuzzy relation, but it is not a transitive fuzzy relation in most cases. In this case, the transitive closure is used to compute a fuzzy equivalence relation from a given fuzzy relation. Given a fuzzy relation $\tilde{R}$, its transitive closure R is computed as

$$R = \tilde{R} \cup \tilde{R}^2 \cup \cdots \cup \tilde{R}^{n-1}.$$

To decrease the complexity, various methods have been proposed to accelerate the computation [6,12,27]. Lee et al. proposed an algorithm for computing the transitive closure of a fuzzy similarity relation that runs in time linearly proportional to the number of elements in the relation, i.e. $O(n^2)$ if the relation (or matrix) is an $n \times n$ relation (or matrix) [12]. In view of its low complexity and easy realization, we use this to compute the transitive closure R of

$\tilde{R}$ in this paper. For example, in Table 1 $\tilde{R}$ can be calculated as

$$\tilde{R} = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{array} \begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ \end{array}$$

$$\tilde{R} = \left(\begin{array}{c|cccccccc} & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_1 & 1 & 0.33 & 0.14 & 0.20 & 0.40 & 0.60 & 0.75 & 0.17 \\ a_2 & 0.33 & 1 & 0.25 & 0.60 & 0.50 & 0.67 & 0.50 & 0.50 \\ a_3 & 0.14 & 0.25 & 1 & 0.33 & 0.29 & 0.43 & 0.29 & 0.50 \\ a_4 & 0.20 & 0.60 & 0.33 & 1 & 0.40 & 0.60 & 0.40 & 0.40 \\ a_5 & 0.40 & 0.50 & 0.29 & 0.40 & 1 & 0.50 & 0.60 & 0.33 \\ a_6 & 0.60 & 0.67 & 0.43 & 0.60 & 0.50 & 1 & 0.80 & 0.50 \\ a_7 & 0.75 & 0.50 & 0.29 & 0.40 & 0.60 & 0.80 & 1 & 0.33 \\ a_8 & 0.17 & 0.50 & 0.50 & 0.40 & 0.33 & 0.50 & 0.33 & 1 \end{array} \right)$$

and by the transitive closure algorithm [12] the fuzzy equivalence relation R induced from $\tilde{R}$ is

$$R = \begin{pmatrix} 1 & 0.67 & 0.50 & 0.60 & 0.60 & 0.75 & 0.75 & 0.50 \\ 0.67 & 1 & 0.50 & 0.60 & 0.60 & 0.67 & 0.67 & 0.50 \\ 0.50 & 0.50 & 1 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.60 & 0.60 & 0.50 & 1 & 0.60 & 0.60 & 0.60 & 0.40 \\ 0.60 & 0.60 & 0.50 & 0.60 & 1 & 0.60 & 0.60 & 0.50 \\ 0.75 & 0.67 & 0.50 & 0.60 & 0.60 & 1 & 0.80 & 0.50 \\ 0.75 & 0.67 & 0.50 & 0.60 & 0.60 & 0.80 & 1 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.40 & 0.50 & 0.50 & 0.50 & 1 \end{pmatrix}.$$

To transform a fuzzy equivalence relation to crisp equivalence relations, a parameter $\delta \in [0, 1]$ is introduced. Let $\delta \in [0, 1]$. If $R \in \mathscr{F}(M \times M)$ is a fuzzy equivalence relation, then the corresponding δ-cut equivalence relation R_δ can be defined as

$$R_\delta = (\tilde{r}_{ij}) \quad \text{where } \tilde{r}_{ij} = \begin{cases} 1, & r_{ij} \geq \delta, \\ 0, & r_{ij} < \delta. \end{cases}$$

There are some important corresponding properties. For example, R_δ is still an equivalence relation and M/R_δ is a partition.

It is important to study FCA in the light of GrC, which has theoretical significance and application potential. One of the strengths of GrC is that larger granulations help in identifying useful information and hide some specific details, and the problem can be solved from the overall picture. Although GrC has been widely explored for in data mining, studies and applications of GrC in FCA are few at present, so further efforts are still needed. In this paper, GrC is introduced into a fuzzy granularity base, and FCA based on this fuzzy granularity base for different granulations is proposed. An axiomatic definition of the granulation of a fuzzy granularity base with $\delta \in [0, 1]$ is given [14]. Let $M/R_\delta = \{P_1, P_2, \ldots, P_m\}$. Then the granulation of R_δ can be defined as

$$\rho(R_\delta) = \frac{1}{|M|^2} \cdot \sum_{i=1}^{m} |P_i|^2.$$

$P_i \in M/R_\delta$ is called a granule. $\Sigma = (M, R)$ with the granulation $\rho(R_\delta)$ is denoted as $\Sigma_\delta = (M_\delta, R_\delta)$, where $M_\delta = M/R_\delta$.

Note that in this paper, any two attributes in the same granule are indistinguishable; thus, different attributes in one granule can be viewed as the same attribute in essence. Any set $\{p_1, p_2, \ldots, p_n\}$ is abbreviated to $p_1 p_2 \ldots p_n$ in some of the following tables.

For any $x \in M_\delta$, if there always exists $y \in M_\tau$ such that $x \subseteq y$, we say M_δ is a refinement of M_τ and M_τ is a coarsening of M_δ, which is denoted as $M_\delta \preceq M_\tau$. That is, by changing the parameter δ, we can obtain $\Sigma = (M, R)$ with different granulations.

Table 2
Σ with different granulations.

R_δ	$\rho(R_\delta)$	$\lvert M_\delta \rvert$	M_δ
$R_{0.50}$	1.00	1	$\{a_1a_2a_3a_4a_5a_6a_7a_8\}$
$R_{0.60}$	0.63	2	$\{a_1a_2a_4a_5a_6a_7, a_3a_8\}$
$R_{0.67}$	0.31	5	$\{a_1a_2a_6a_7, a_3, a_4, a_5, a_8\}$
$R_{0.75}$	0.22	6	$\{a_1a_6a_7, a_2, a_3, a_4, a_5, a_8\}$
$R_{0.80}$	0.16	7	$\{a_1, a_2, a_3, a_4, a_5, a_6a_7, a_8\}$
$R_{1.00}$	0.13	8	$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$

For example, from Table 2 (induced from Table 1) some conclusions can be immediately drawn.

- Let $\delta, \tau \in [0, 1]$. If $\tau \le \delta$, then $\rho(R_\delta) \le \rho(R_\tau)$ and $M_\delta \preceq M_\tau$.
- Let $\delta \in [0, 1]$. When $\rho(R_\delta)$ decreases, $|M_\delta|$ increases; conversely, when $\rho(R_\delta)$ increases, $|M_\delta|$ decreases.

For convenient description, some formal symbols used in the tables are defined. a_i is denoted as α_i, where $1 \le i \le 8$. $a_1a_2a_6a_7$, a_3a_8 and $a_1a_2a_4a_5a_6a_7$ are denoted as α_9, α_{10} and α_{11}, respectively.

4. δ-Concept lattice and δ-rule

Major issues that arise for concept lattices are their high space complexity and time complexity. In particular, for dense and large contexts, the performance of most algorithms is not ideal. Efficient generation of a concept lattice for large-scale data has become the primary problem in FCA applications. To address this problem, a new structure called δ-concept lattice is designed to reduce the scale of nodes and the time complexity in generating a concept lattice. In this section we build a δ-concept lattice based on fuzzy granularity with the granulation $\rho(R_\delta)$, and extract a complete and nonredundant set of δ-rules from the δ-concept.

A formal context $K = (G, M, I)$ with the granulation $\rho(R_\delta)$ is a quadruplet $K_\delta = (G, M, I, M_\delta)$. For example, Table 3 is a formal context with the granulation $\rho(R_{0.67})$ induced from Table 1.

In $K_\delta = (G, M, I, M_\delta)$, for a set $A \subseteq G$ we define

$$A' = \{\alpha \in M_\delta | xIm \text{ and } \exists m \in \alpha \text{ for all } x \in A\}.$$

Correspondingly, for a set $\widetilde{B} \subseteq M_\delta$ we define

$$\widetilde{B}' = \{x \in G | xIm \text{ and } \exists m \in \alpha \text{ for all } \alpha \in \widetilde{B}\}.$$

Theorem 1. *In K_δ, if $A, A_1, A_2 \subseteq G$ and $\widetilde{B}, \widetilde{B}_1, \widetilde{B}_2 \subseteq M_\delta$, then*

$$(1)\ A_1 \subseteq A_2 \Rightarrow A'_2 \subseteq A'_1 \quad (2)\ \widetilde{B}_1 \subseteq \widetilde{B}_2 \Rightarrow \widetilde{B}'_2 \subseteq \widetilde{B}'_1,$$
$$(3)\ A \subseteq A'';\ \widetilde{B} \subseteq \widetilde{B}'' \quad (4)\ A' = A''';\ \widetilde{B}' = \widetilde{B}'''.$$

Theorem 2. *In K_δ, for every $s \in S$, if $A_s \subseteq G$ and $\widetilde{B}_s \subseteq M_\delta$, then*

$$(1)\ \left(\bigcup_{s\in S} A_s\right)' = \bigcap_{s\in S} A_s{}' \quad (2)\ \left(\bigcup_{s\in S} \widetilde{B}_s\right)' = \bigcap_{s\in S} \widetilde{B}'_s,$$

where S is an index set.

In $K_\delta = (G, M, I, M_\delta)$, let $A \subseteq G$ and $\widetilde{B} \subseteq M_\delta$. If $A' = \widetilde{B}$ and $\widetilde{B}' = A$, we say $(A, \widetilde{B})$ is a δ-concept. The set of all δ-concepts is denoted as $\mathscr{B}(K_\delta)$. Let $(A_1, \widetilde{B}_1)$ and $(A_2, \widetilde{B}_2)$ be two δ-concepts. We define

$$(A_1, \widetilde{B}_1) \preceq_\delta (A_2, \widetilde{B}_2) \;\Leftrightarrow\; A_1 \subseteq A_2 \;\Leftrightarrow\; \widetilde{B}_1 \supseteq \widetilde{B}_2.$$

Table 3
A formal context with the granulation $\rho(R_{0.67})$.

G	α_9				α_3	α_4	α_5	α_8
	a_1	a_2	a_6	a_7	a_3	a_4	a_5	a_8
1	×	×	×	×			×	×
2					×			×
3	×		×	×	×			
4		×						
5		×	×	×	×	×	×	×
6					×		×	
7		×	×		×	×		×
8	×	×	×	×		×	×	

The relation "$\preceq_\delta$" is the hierarchical order of δ-concepts. Obviously, a lattice structure can be deduced that is a complete lattice. This is called a δ-concept lattice and is described in the following theorem. It is still denoted as $\mathscr{B}(K_\delta)$ if there is no danger of confusion.

Theorem 3. *In K_δ, the partially ordered set $\mathscr{B}(K_\delta)$ is a complete lattice, where the corresponding infimum and supremum are defined as*

$$\bigwedge_{s\in S}(A_s, \widetilde{B}_s) = \left(\bigcap_{s\in S} A_s, \left(\bigcup_{s\in S} \widetilde{B}_s\right)''\right)$$

$$\bigvee_{s\in S}(A_s, \widetilde{B}_s) = \left(\left(\bigcup_{s\in S} A_s\right)'', \bigcap_{s\in S} \widetilde{B}_s\right).$$

Comparison reveals many similarities and some slight differences between the model proposed in this paper and the α Galois lattice of Ventos and Soldano [26], which are two entirely different models. Equivalence relations and parameters are introduced into the two models, but their effects are different. For example, R_δ in this paper is an equivalence relation on the set of attributes, while r in the α Galois lattice is an equivalence relation on the set of objects. The parameter δ in this paper is introduced into a fuzzy equivalence relation, while α in the α Galois lattice is not.

The steps for generating a δ-concept lattice are now described, but detailed descriptions are omitted.

1. Input $\delta \in [0, 1]$ and then generate $M_\delta = M/R_\delta$.
2. In $K_\delta = (G, M, I, M_\delta)$, because any $\alpha \in M_\delta$ can be viewed as an attribute, a δ-concept lattice can be generated using traditional algorithms for building concept lattices with little change.
3. Output the δ-concept lattice.

For a traditional formal context $K = (G, M, I)$ with $|M| = n$, the upper bound on the number of nodes in the corresponding lattice structure is 2^n, which seems a serious impediment to application of concept lattices. Although a formal concept with upper bound 2^n is rare in the real world, the upper bound of common formal contexts can still be reached

$$N = 2 + C_n^1 + C_n^2 + \cdots + C_n^k$$

(suppose each object in K has k attributes at most). Similarly, for $K_\delta = (G, M, I, M_\delta)$ with $|M_\delta| = m$ we can obtain the upper bound

$$N_\delta = 2 + C_m^1 + C_m^2 + \cdots + C_m^l$$

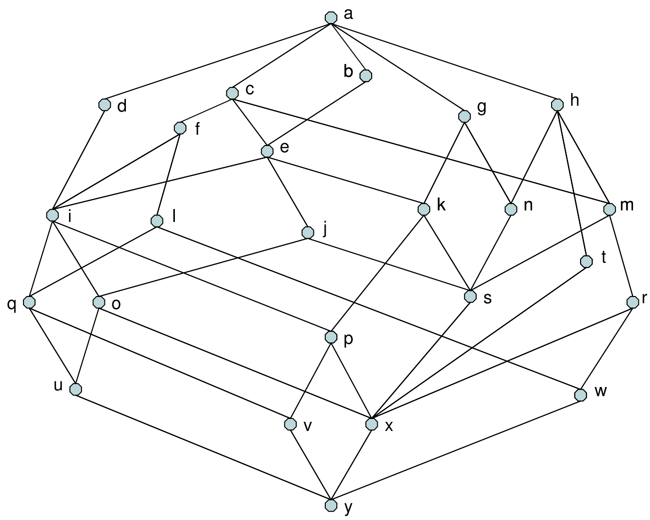

Fig. 1. $\mathscr{B}(K_1)$.

(suppose each object g in K_δ satisfies $|\{g\}'|\le l$). Furthermore, we can draw the following conclusions:

- When the value of δ increases, the value of N_δ also increases. In particular, when $\delta = 1$, then $N_\delta = 2 + C_n^1 + C_n^2 + \cdots + C_n^k$.
- When the value of δ decreases, the value of N_δ also decreases. In particular, when $\delta = 0$, then $N_\delta = 2$.

Currently, to generate a concept lattice from (G, M, I) in the worst case, although the minimum time complexity of some algorithms is $O((|G| + |M|) \cdot |M| \cdot |L|)$ (where $|L|$ is the number of nodes in the concept lattice and there is an exponential relationship between $|L|$ and $|M|$), the time complexity of most algorithms is $O(|G| \cdot |M|^2 \cdot |L|)$. For the completeness of concept lattices, identification of a construction algorithm with much better time complexity than existing algorithms seems impossible. Thus, efficient lattice construction from a large context is still a key issue in FCA research. We can obtain the time complexity for generating $\mathscr{B}(K_\delta)$ as follows. In $K_\delta = (G, M, I, M_\delta)$, we know the time complexity for generating M_δ will be $O(2 \cdot |M|^2)$; the time complexity for generating a δ-concept lattice will be $O((|G| + |M_\delta|) \cdot |M_\delta| \cdot |L_\delta|)$ (where $|L_\delta|$ is the number of nodes in $\mathscr{B}(K_\delta)$ and there is an exponential relationship between $|L_\delta|$ and $|M_\delta|$). Thus, the time complexity for generating $\mathscr{B}(K_\delta)$ will be

$$O(2 \cdot |M|^2 + (|G| + |M_\delta|) \cdot |M_\delta| \cdot |L_\delta|).$$

We can obtain the following conclusions.

- When the value of δ increases, the time complexity for generating $\mathscr{B}(K_\delta)$ also increases and converges to $O((|G| + |M_\delta|) \cdot |M_\delta| \cdot |L_\delta|)$. In particular, when $\delta = 1$, the time complexity is $O((|G| + |M|) \cdot |M| \cdot |L|)$.
- When the value of δ decreases, the time complexity for generating $\mathscr{B}(K_\delta)$ also decreases and converges to $O(2 \cdot |M|^2)$. In particular, when $\delta = 0$, the time complexity is $O(2 \cdot |M|^2)$.

The above analysis indicates that the time complexity can be reduced to some extent, but this is not necessarily good, especially for larger experiments. Thus, research is still valuable because we strongly believe that GrC is key to reducing the time complexity and the scale of nodes for efficient generation of a concept lattice.

We illustrate the above method for generating a δ-concept lattice with an example. For Table 3, $\mathscr{B}(K_{0.67})$ is depicted in Fig. 2. Table 5 is a specific interpretation of all nodes in Fig. 2. For $\mathscr{B}(K_1)$ depicted in Fig. 1 and $\mathscr{B}(K_{0.6})$ in Fig. 3, the nodes are explained in Tables 4 and 6, respectively.

From the above illustration it is evident that when the granulation $\rho(R_\delta)$ decreases, the size of the δ-concept lattice increases. Conversely, when the granularity $\rho(R_\delta)$ increases, the size of the δ-concept lattice decreases. In practical applications, by changing the parameter δ, users can build δ-concepts to meet their needs.

Many researchers have studied the problem of rule extraction from concept lattices. Compared with other methods, rules extracted from a concept lattice have equal or better effects. Considering the situation mentioned above, we

Table 4
Specific interpretation of all the nodes in Fig. 1.

a	$(12345678,\ \emptyset)$	j	$(578,\ \alpha_2\alpha_4\alpha_6)$	s	$(57,\ \alpha_2\alpha_3\alpha_4\alpha_6\alpha_8)$
b	$(14578,\ \alpha_2)$	k	$(157,\ \alpha_2\alpha_6\alpha_8)$	t	$(56,\ \alpha_3\alpha_5)$
c	$(13578,\ \alpha_6)$	l	$(138,\ \alpha_1\alpha_6\alpha_7)$	u	$(8,\ \alpha_1\alpha_2\alpha_4\alpha_5\alpha_6\alpha_7)$
d	$(1568,\ \alpha_5)$	m	$(357,\ \alpha_3\alpha_6)$	v	$(1,\ \alpha_1\alpha_2\alpha_5\alpha_6\alpha_7\alpha_8)$
e	$(1578,\ \alpha_2\alpha_6)$	n	$(257,\ \alpha_3\alpha_8)$	w	$(3,\ \alpha_1\alpha_3\alpha_6\alpha_7)$
f	$(1358,\ \alpha_6\alpha_7)$	o	$(58,\ \alpha_2\alpha_4\alpha_5\alpha_6\alpha_7)$	x	$(5,\ \alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8)$
g	$(1257,\ \alpha_8)$	p	$(15,\ \alpha_2\alpha_5\alpha_6\alpha_7\alpha_8)$	y	$(\emptyset,\ \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8)$
h	$(23567,\ \alpha_3)$	q	$(18,\ \alpha_1\alpha_2\alpha_5\alpha_6\alpha_7)$		
i	$(158,\ \alpha_2\alpha_5\alpha_6\alpha_7)$	r	$(35,\ \alpha_3\alpha_6\alpha_7)$		

Table 5
Specific interpretation of all the nodes in Fig. 2.

$\underline{a}$	$(12345678,\ \emptyset)$	$\underline{f}$	$(257,\ \alpha_3\alpha_8)$	$\underline{k}$	$(15,\ \alpha_9\alpha_5\alpha_8)$
$\underline{b}$	$(1257,\ \alpha_8)$	$\underline{g}$	$(157,\ \alpha_9\alpha_8)$	$\underline{l}$	$(56,\ \alpha_3\alpha_5)$
$\underline{c}$	$(23567,\ \alpha_3)$	$\underline{h}$	$(357,\ \alpha_9\alpha_3)$	$\underline{m}$	$(57,\ \alpha_9\alpha_3\alpha_4\alpha_8)$
$\underline{d}$	$(1568,\ \alpha_5)$	$\underline{i}$	$(158,\ \alpha_9\alpha_5)$	$\underline{n}$	$(58,\ \alpha_9\alpha_4\alpha_5)$
$\underline{e}$	$(134578,\ \alpha_9)$	$\underline{j}$	$(578,\ \alpha_9\alpha_4)$	$\underline{o}$	$(5,\ \alpha_9\alpha_3\alpha_4\alpha_5\alpha_8)$

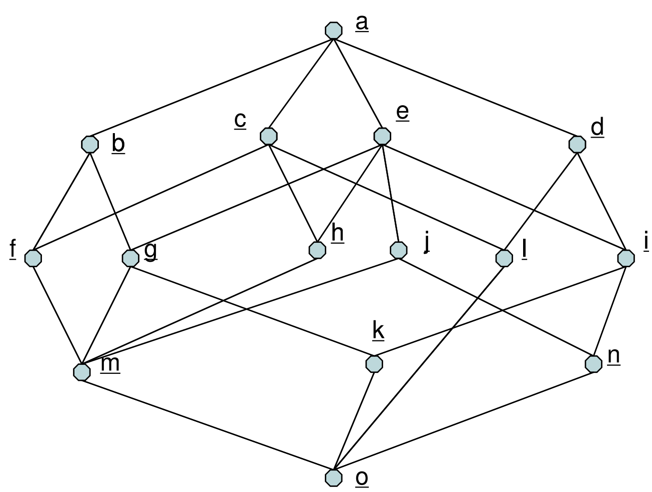

Fig. 2. $\mathscr{B}(K_{0.67})$.

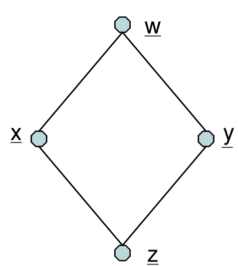

Fig. 3. $\mathscr{B}(K_{0.6})$.

Table 6
Specific interpretation of all the nodes in Fig. 3.

$\underline{w}$	$(12345678,\ \emptyset)$	$\underline{y}$	$(1345678,\ \alpha_{11})$
$\underline{x}$	$(123567,\ \alpha_{10})$	$\underline{z}$	$(13567,\ \alpha_{10}\alpha_{11})$

Table 7
A complete and nonredundant set of 1-rules.

$\alpha_1 \xrightarrow{1} \alpha_6\alpha_7$	$\alpha_1\alpha_2\alpha_6\alpha_7 \xrightarrow{1} \alpha_5$	$\alpha_2\alpha_4\alpha_5\alpha_6 \xrightarrow{1} \alpha_7$
$\alpha_4 \xrightarrow{1} \alpha_2\alpha_6$	$\alpha_1\alpha_2\alpha_3\alpha_5\alpha_6\alpha_7 \xrightarrow{1} \alpha_4\alpha_8$	$\alpha_2\alpha_4\alpha_6\alpha_7 \xrightarrow{1} \alpha_5$
$\alpha_7 \xrightarrow{1} \alpha_6$	$\alpha_1\alpha_2\alpha_3\alpha_5\alpha_6\alpha_7\alpha_8 \xrightarrow{1} \alpha_4$	$\alpha_2\alpha_4\alpha_6\alpha_8 \xrightarrow{1} \alpha_3$
$\alpha_5\alpha_8 \xrightarrow{1} \alpha_2\alpha_6\alpha_7$	$\alpha_2\alpha_3\alpha_6\alpha_8 \xrightarrow{1} \alpha_4$	$\alpha_5\alpha_6\alpha_7 \xrightarrow{1} \alpha_2$
$\alpha_6\alpha_8 \xrightarrow{1} \alpha_2$	$\alpha_1\alpha_2\alpha_4\alpha_6\alpha_7 \xrightarrow{1} \alpha_5$	$\alpha_6\alpha_7\alpha_8 \xrightarrow{1} \alpha_2\alpha_5$
$\alpha_5\alpha_6 \xrightarrow{1} \alpha_2\alpha_7$	$\alpha_1\alpha_5\alpha_6\alpha_7 \xrightarrow{1} \alpha_2$	$\alpha_2\alpha_3\alpha_5\alpha_6\alpha_7 \xrightarrow{1} \alpha_4\alpha_8$
$\alpha_2\alpha_8 \xrightarrow{1} \alpha_6$	$\alpha_1\alpha_6\alpha_7\alpha_8 \xrightarrow{1} \alpha_2\alpha_5$	$\alpha_2\alpha_3\alpha_5\alpha_6\alpha_7\alpha_8 \xrightarrow{1} \alpha_4$
$\alpha_2\alpha_5 \xrightarrow{1} \alpha_6\alpha_7$	$\alpha_2\alpha_6\alpha_7 \xrightarrow{1} \alpha_5$	
$\alpha_2\alpha_3 \xrightarrow{1} \alpha_4\alpha_6\alpha_8$	$\alpha_2\alpha_3\alpha_4\alpha_6 \xrightarrow{1} \alpha_8$	

Table 8
A complete and nonredundant set of 0.67-rules.

$\alpha_4 \xrightarrow{0.67} \alpha_9$	$\alpha_9\alpha_4\alpha_8 \xrightarrow{0.67} \alpha_3$	$\alpha_9\alpha_5 \xrightarrow{0.67} \alpha_4\alpha_8$	$\alpha_9\alpha_3\alpha_5\alpha_8 \xrightarrow{0.67} \alpha_4$
$\alpha_9\alpha_3\alpha_4 \xrightarrow{0.67} \alpha_8$	$\alpha_5\alpha_8 \xrightarrow{0.67} \alpha_9$	$\alpha_9\alpha_3\alpha_8 \xrightarrow{0.67} \alpha_4$	

investigate δ-rules extracted from δ-concept lattices and obtain some significant conclusions. By adopting the method in [8] with little changes, we can extract δ-rules and obtain a complete and nonredundant set of δ-rules, where a δ-rule is defined as follows.

In $K_\delta = (G, M, I, M_\delta)$, let $\widetilde{B}_1, \widetilde{B}_2 \subseteq M_\delta$. If $\widetilde{B}_1' \subseteq \widetilde{B}_2'$, we say that $\widetilde{B}_1 \xrightarrow{\delta} \widetilde{B}_2$ is a δ-rule. For example, from Fig. 1 a complete and nonredundant set of 1-rules can be extracted, as shown in Table 7, and from Fig. 2 a complete and nonredundant set of 0.67-rules can be extracted, as shown in Table 8. The number of δ-rules is much less in Table 8 than in Table 7. In fact, as the granulation $\rho(R_\delta)$ increases, the number of δ-rules will obviously decrease. Moreover, in a formal context $K = (G, M, I)$, if there are no identical columns, then some conclusions can be inferred immediately, as follows.

- There exists a bijection $\varphi:\ M \to M_1$.
- If $(A, B) \in \mathscr{B}(K)$, then $(A, \varphi(B)) \in \mathscr{B}(K_1)$.
- $\mathscr{B}(K) \cong \mathscr{B}(K_1)$.
- $\varphi(C_1) \xrightarrow{1} \varphi(C_2)$ is equal to $C_1 \to C_2$, where $C_1, C_2 \subseteq M$.

In other words, $\mathscr{B}(K)$ is just a special case of $\mathscr{B}(K_\delta)$ and $C_1 \to C_2$ is just a special case of $\varphi(C_1) \xrightarrow{\delta} \varphi(C_2)$.

5. (θ, σ)-Rules

In decision contexts, decision rules have become a common form of knowledge representation in practice because of their expressiveness and ease of understanding. In this section, we discuss decision rules called (θ, σ)-rules.

A formal context $K = (G, M, I)$ is called a decision context if $M = B \cup D$ and $I = I^B \cup I^D$, where B is the set of condition attributes, D is the set of decision attributes, $I^B = I \cap (G \times B)$ and $I^D = I \cap (G \times D)$.

The decision context shown in Table 9 is for a grocery shop. The set of objects G is composed of $1, 2, \ldots, 8$ denoting customer codes, the set of attributes B is composed of $a_1, a_2, \ldots, a_8$ denoting commodity codes, and the set D of

Table 9
A sample data set.

G	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	d_1	d_2	d_3	d_4	d_5
1	×	×			×	×	×	×	×		×	×	
2			×					×	×	×		×	
3	×		×			×	×		×	×			×
4		×							×	×			×
5		×	×	×	×	×	×	×			×	×	×
6			×		×				×	×	×		×
7		×	×	×		×		×			×	×	
8	×	×		×	×	×	×				×	×	

Table 10
A (0.67,0.8)-decision context.

G	α_9				α_3	α_4	α_5	α_8	β_6		β_3	β_4	β_5
	a_1	a_2	a_6	a_7	a_3	a_4	a_5	a_8	d_1	d_2	d_3	d_4	d_5
1	×	×	×	×			×	×	×		×	×	
2					×			×	×	×		×	
3	×		×	×	×				×	×			×
4		×							×	×			×
5		×	×	×	×	×	×	×			×	×	×
6					×		×		×	×	×		×
7		×	×		×	×		×			×	×	
8	×	×	×	×		×	×				×	×	

decision attributes is composed of $d_1, d_2, \ldots, d_5$, also denoting commodity codes. If customer i purchases commodity a_j or d_k, this is denoted as $(i, a_j) \in I_B$ or $(i, d_k) \in I_D$ and is shown as "×" in the table.

A decision context consists of two subcontexts, the condition subcontext $K^B = (G, B, I^B)$ and the decision subcontext $K^D = (G, D, I^D)$. The fuzzy granularity base of K^B is denoted as $\Sigma^B = (B, R^B)$, while the fuzzy granularity base of K^D is denoted as $\Sigma^D = (D, R^D)$. Σ^B with the granulation $\rho(R^B_\theta)$ is denoted as $\Sigma^B_\theta = (B_\theta, R^B_\theta)$, where $B_\theta = B/R^B_\theta$. Similarly, Σ^D with the granulation $\rho(R^D_\sigma)$ is denoted as $\Sigma^D_\sigma = (D_\sigma, R^D_\sigma)$, where $D_\sigma = D/R^D_\sigma$.

Let $\theta, \sigma \in [0, 1]$. If K^B_θ is a subcontext with the granulation $\rho(R^B_\theta)$ and K^D_σ is a subcontext with the granulation $\rho(R^D_\sigma)$, we say that K is a (θ, σ)-decision context. Table 10 shows a (θ, σ)-decision context.

For convenience, some formal symbols for a (θ, σ)-decision context are defined as follows. For $A \subseteq G$, A' in K^B_θ is denoted as A^{I_1} and A' in K^D_σ as A^{I_2}; correspondingly, for $\widetilde{B}_1 \subseteq B_\theta$, $\widetilde{B}'_1$ in K^B_θ is denoted as $\widetilde{B}^{I_1}_1$, and for $\widetilde{D}_1 \subseteq D_\sigma$, $\widetilde{D}'_1$ in K^D_σ is denoted as $\widetilde{D}^{I_2}_1$. The set of all intents of θ-concepts in $\mathscr{B}(K^B_\theta)$ is denoted as $\mathscr{U}^B_\theta$ and the set of all intents of σ-concepts in $\mathscr{B}(K^D_\sigma)$ is denoted as $\mathscr{U}^D_\sigma$.

Theorem 4. *In a (θ, σ)-decision context, let $\widetilde{B}_1 \subseteq B_\theta$ and $\widetilde{D}_1 \subseteq D_\sigma$. Then*

$$(1)\ \widetilde{B}^{I_1 I_1}_1 = (\widetilde{B}_1)^+_\theta \quad (2)\ \widetilde{D}^{I_2 I_2}_1 = (\widetilde{D}_1)^+_\sigma,$$

where $(\widetilde{B}_1)^+_\theta = \bigcap\{\widetilde{T} \in \mathscr{U}^B_\theta | \widetilde{B}_1 \subseteq \widetilde{T}\}$ *and* $(\widetilde{D}_1)^+_\sigma = \bigcap\{\widetilde{T} \in \mathscr{U}^D_\sigma | \widetilde{D}_1 \subseteq \widetilde{T}\}$.

In a (θ, σ)-decision context, let $\widetilde{B}_1 \subseteq B_\theta$ and $\widetilde{D}_1 \subseteq D_\sigma$. If $\widetilde{B}^{I_1}_1 \subseteq \widetilde{D}^{I_2}_1$, then we say that $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ is a (θ, σ)-rule; if $\widetilde{B}_1 = (\widetilde{B}_1)^+_\theta$ and $\widetilde{D}_1 = (\widetilde{D}_1)^+_\sigma$, then we say that the (θ, σ)-rule $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ is a (θ, σ)-concept rule.

Theorem 5. *In a (θ, σ)-decision context, let $\widetilde{B}_1, \widetilde{B}_2 \subseteq B_\theta$ and $\widetilde{D}_1, \widetilde{D}_2 \subseteq D_\sigma$. If $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ is a (θ, σ)-concept rule, then*:

(1) *If $\widetilde{B}_1 \subseteq \widetilde{B}_2$ and $\widetilde{D}_2 \subseteq \widetilde{D}_1$, then $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$.*
(2) *If $\widetilde{B}_1 = (\widetilde{B}_2)^+_\theta$ and $\widetilde{D}_1 = (\widetilde{D}_2)^+_\sigma$, then $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$.*

Proof. (1) From $\widetilde{B}_1 \subseteq \widetilde{B}_2$ and $\widetilde{D}_2 \subseteq \widetilde{D}_1$, it follows that $\widetilde{B}_2^{I_1} \subseteq \widetilde{B}_1^{I_1}$ and $\widetilde{D}_1^{I_2} \subseteq \widetilde{D}_2^{I_2}$. Then we can obtain $\widetilde{B}_2^{I_1} \subseteq \widetilde{D}_2^{I_2}$ based on $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1 \Rightarrow \widetilde{B}_1^{I_1} \subseteq \widetilde{D}_1^{I_2}$. Hence, $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ holds.

(2) From $\widetilde{B}_1 = (\widetilde{B}_2)^+_\theta$ and $\widetilde{D}_1 = (\widetilde{D}_2)^+_\sigma$, it follows that $\widetilde{B}_2^{I_1 I_1} = \widetilde{B}_1$ and $\widetilde{D}_2^{I_2 I_2} = \widetilde{D}_1$. Then $\widetilde{B}_2^{I_1} = \widetilde{B}_1^{I_1}$ and $\widetilde{D}_2^{I_2} = \widetilde{D}_1^{I_2}$. In addition, $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1 \Rightarrow \widetilde{B}_1^{I_1} \subseteq \widetilde{D}_1^{I_2}$. Hence, $\widetilde{B}_2^{I_1} \subseteq \widetilde{D}_2^{I_2}$; that is, $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ holds. □

In general, the number of (θ, σ)-rules in a decision context is quite large, and in a given set of (θ, σ)-rules there are many redundant (θ, σ)-rules that can be deduced from others by means of so-called inference rules.

(θ, σ)-*Decision inference*: In a (θ, σ)-decision context, let $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ be a (θ, σ)-concept rule and let $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ be a (θ, σ)-rule. Then:

- ((θ, σ)-inference rule 1) If $\widetilde{B}_1 \subseteq \widetilde{B}_2$, $\widetilde{D}_2 \subseteq \widetilde{D}_1$ and $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ is a (θ, σ)-concept rule, then $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ can be inferred from $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$.
- ((θ, σ)-inference rule 2) If $\widetilde{B}_1 = (\widetilde{B}_2)^+_\theta$ and $\widetilde{D}_1 = (\widetilde{D}_2)^+_\sigma$, then $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ can be inferred from $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$.

(θ, σ)-Inference rule 1 takes the following form:

$$\frac{\widetilde{B}_1 \subseteq \widetilde{B}_2,\ \widetilde{D}_2 \subseteq \widetilde{D}_1,\ (\theta,\sigma)\text{-concept rule } \widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1}{(\theta,\sigma)\text{-concept rule } \widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2},$$

which means that if $\widetilde{B}_1 \subseteq \widetilde{B}_2$, $\widetilde{D}_2 \subseteq \widetilde{D}_1$ and $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ is a (θ, σ)-concept rule, then a (θ, σ)-concept rule $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ can be inferred. Correspondingly, (θ, σ)-inference rule 2 takes the form

$$\frac{\widetilde{B}_1 = (\widetilde{B}_2)^+_\theta,\ \widetilde{D}_1 = (\widetilde{D}_2)^+_\sigma,\ (\theta,\sigma)\text{-concept rule } \widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1}{\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2}.$$

That is, if $\widetilde{B}_1 = (\widetilde{B}_2)^+_\theta$, $\widetilde{D}_1 = (\widetilde{D}_2)^+_\sigma$ and $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ is a (θ, σ)-concept rule, then a (θ, σ)-rule $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ can be inferred.

In a (θ, σ)-decision context, let $\Delta_{(\theta,\sigma)}$ be a set of (θ, σ)-rules and let $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ be a (θ, σ)-rule. If $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ can be inferred from $\Delta_{(\theta,\sigma)}$ by some decision inference τ, we say that $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ can be τ-inferred from $\Delta_{(\theta,\sigma)}$. In this case, we call $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ redundant relative to $\Delta_{(\theta,\sigma)}$. If all (θ, σ)-rules can be τ-inferred from $\Delta_{(\theta,\sigma)}$, we say that $\Delta_{(\theta,\sigma)}$ is τ-complete. Moreover, a (θ, σ)-concept rule $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ is maximal if:

(1) There is no (θ, σ)-concept rule $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ satisfying $\widetilde{B}_2 \subset \widetilde{B}_1$.
(2) There is no (θ, σ)-concept rule $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ satisfying $\widetilde{D}_1 \subset \widetilde{D}_2$.

Theorem 6. *In a (θ, σ)-decision context, if $\Delta_{(\theta,\sigma)}$ is the set of all maximal (θ, σ)-concept rules and τ is the (θ, σ)-decision inference, then $\Delta_{(\theta,\sigma)}$ is τ-nonredundant and τ-complete.*

Proof. First, we prove that $\Delta_{(\theta,\sigma)}$ is τ-nonredundant. If we assume that $\Delta_{(\theta,\sigma)}$ is τ-redundant, then there must exist a maximal (θ, σ)-concept rule $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$, which can be inferred from $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ in $\Delta_{(\theta,\sigma)} \backslash (\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1)$ based on (θ,σ)-inference rule 1. Obviously, we have $\widetilde{B}_2 \subseteq \widetilde{B}_1$ and $\widetilde{D}_1 \subseteq \widetilde{D}_2$. In addition, since $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1 \notin \Delta_{(\theta,\sigma)} \backslash (\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1)$, there must exist $\widetilde{B}_1 \neq \widetilde{B}_2$ or $\widetilde{D}_1 \neq \widetilde{D}_2$.

If $\widetilde{D}_1 \neq \widetilde{D}_2$, then $\widetilde{D}_1 \subset \widetilde{D}_2$. Since $\widetilde{B}_2 \subseteq \widetilde{B}_1$, $\widetilde{B}_2^{I_1} \supseteq \widetilde{B}_1^{I_1}$ holds. In addition, we can obtain $\widetilde{B}_2^{I_1} \subseteq \widetilde{D}_2^{I_2}$ by $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$. Furthermore, $\widetilde{B}_1^{I_1} \subseteq \widetilde{D}_2^{I_2}$; that is, $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ holds. Obviously, $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ contradicts the condition whereby $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ is a maximal (θ,σ)-concept rule.

When $\widetilde{D}_1 = \widetilde{D}_2$ and $\widetilde{B}_1 \neq \widetilde{B}_2$, because $\widetilde{B}_2 \subseteq \widetilde{B}_1$, then $\widetilde{B}_2 \subset \widetilde{B}_1$. Since $\widetilde{D}_1 = \widetilde{D}_2$, there exists $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$, which contradicts the condition whereby $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ is a maximal (θ,σ)-concept rule. Hence, it is clear that $\Delta_{(\theta,\sigma)}$ is τ-nonredundant from the above discussion.

Next, we prove that $\Delta_{(\theta,\sigma)}$ is τ-complete. Let $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2 \notin \Delta_{(\theta,\sigma)}$ be a (θ,σ)-rule. We can easily see that $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ can be inferred from $(\widetilde{B}_2)_\theta^+ \xrightarrow{(\theta,\sigma)} (\widetilde{D}_2)_\sigma^+$ based on (θ,σ)-inference rule 2. Furthermore, we can see that there must exist a maximal (θ,σ)-concept rule $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$, and that $(\widetilde{B}_2)_\theta^+ \xrightarrow{(\theta,\sigma)} (\widetilde{D}_2)_\sigma^+$ can be inferred from $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$ based on (θ,σ)-inference rule 1. Hence, $\widetilde{B}_2 \xrightarrow{(\theta,\sigma)} \widetilde{D}_2$ can be inferred from $\widetilde{B}_1 \xrightarrow{(\theta,\sigma)} \widetilde{D}_1$. This indicates that $\Delta_{(\theta,\sigma)}$ is τ-complete. $\square$

Although many inference rules can be used to eliminate superfluous (θ,σ)-rules, the (θ,σ)-inference rules proposed here are perhaps more intuitive and simpler in comparison with other inference rules. Based on $\mathscr{B}(K_\theta^B)$ and $\mathscr{B}(K_\sigma^D)$, the steps for generating a τ-complete and τ-nonredundant set $\Delta_{(\theta,\sigma)}$ of (θ,σ)-rules are as follows.

1. For every $(A, \widetilde{E}) \in \mathscr{B}(K_\theta^B)$, find $(C_i, \widetilde{D}_i) \in \mathscr{B}(K_\sigma^D)$, which is the minimal concept satisfying $\widetilde{B} \xrightarrow{(\theta,\sigma)} \widetilde{D}_i$. Then add $(C_i, \widetilde{D}_i)$ to $\mathcal{X}$.
2. Choose $(C_j, \widetilde{D}_j)$ from $\mathcal{X}$ randomly, then find $(A_k, \widetilde{B}_k) \in \mathscr{B}(K_\theta^B)$, which is the maximal concept satisfying $\widetilde{B}_k \xrightarrow{(\theta,\sigma)} \widetilde{D}_j$. Delete $(C_j, \widetilde{D}_j)$ from $\mathcal{X}$.
3. Add $\widetilde{B}_k \xrightarrow{(\theta,\sigma)} \widetilde{D}_j$ to $\Delta_{(\theta,\sigma)}$; if $\mathcal{X} \neq \emptyset$, switch to 2.
4. Output $\Delta_{(\theta,\sigma)}$.

The number of θ-concepts is denoted as p and the number of σ-concepts is denoted as q. Then the time complexity for generating $\Delta_{(\theta,\sigma)}$ will be $O(p \cdot q^2 + p^2 \cdot q)$. This is not desirable, especially for larger experiments. Thus, the algorithm only serves as a basis for opportunities for further development.

For example, the fuzzy granularity base of K^D induced from Table 9 is denoted as $\Sigma^D = (D, R^D)$, where

$$R^D = \begin{pmatrix} 1 & 0.8 & 0.28 & 0.28 & 0.6 \\ 0.8 & 1 & 0.28 & 0.28 & 0.6 \\ 0.28 & 0.28 & 1 & 0.67 & 0.28 \\ 0.28 & 0.28 & 0.67 & 1 & 0.28 \\ 0.6 & 0.6 & 0.28 & 0.28 & 1 \end{pmatrix}.$$

The Σ^D with different granulations is shown in Table 11. Some formal symbols in the following tables are defined for convenience. d_i is denoted as β_i, where $1 \leq i \leq 5$. d_1d_2 and d_3d_4 are denoted as β_6 and β_7, respectively.

In a (θ,σ)-decision context, if Σ^B with the granulation $\rho(R_\theta^B)$ or Σ^D with the granulation $\rho(R_\sigma^D)$ is too fine or too coarse for users, it can be changed by adjusting the parameters θ and σ. Then the set of (θ,σ)-rules satisfying user needs can be obtained. For example, Tables 13 and 14 can be obtained from Table 9 based on Tables 12, 4 and 5.

By Theorem 6, all (θ,σ)-rules can be inferred from $\Delta_{(\theta,\sigma)}$ with the (θ,σ)-decision inference. For example, $\alpha_4\alpha_5\alpha_6\alpha_7 \xrightarrow{(1,1)} \beta_3\beta_4$ is a (1, 1)-rule in Table 9, because $\alpha_2\alpha_4\alpha_5\alpha_6\alpha_7 = (\alpha_4\alpha_5\alpha_6\alpha_7)_1^+$ and $\beta_3\beta_4 = (\beta_3\beta_4)_1^+$, $\alpha_4\alpha_5\alpha_6\alpha_7 \xrightarrow{(1,1)} \beta_3\beta_4$ can be inferred from $\alpha_2\alpha_4\alpha_5\alpha_6\alpha_7 \xrightarrow{(1,1)} \beta_3\beta_4$ based on inference rule 2. In addition, because $\beta_3\beta_4 \subseteq \beta_3\beta_4$ and $\alpha_2\alpha_6 \subseteq \alpha_2\alpha_4\alpha_5\alpha_6\alpha_7$, $\alpha_2\alpha_4\alpha_5\alpha_6\alpha_7 \xrightarrow{(1,1)} \beta_3\beta_4$ can be inferred from $\alpha_2\alpha_6 \xrightarrow{(\theta,\sigma)} \beta_3\beta_4$ based on

Table 11
Σ^D with different granulations.

R_σ^D	$\rho(D_\sigma)$	$\|D_\sigma\|$	D_σ
$R_{0.28}$	1.00	1	$\{d_1d_2d_3d_4d_5\}$
$R_{0.60}$	0.52	2	$\{d_1d_2d_5, d_3d_4\}$
$R_{0.67}$	0.36	3	$\{d_1d_2, d_3d_4, d_5\}$
$R_{0.80}$	0.28	4	$\{d_1d_2, d_3, d_4, d_5\}$
$R_{1.00}$	0.20	5	$\{d_1, d_2, d_3, d_4, d_5\}$

Table 12
$\mathscr{B}(K_\sigma^D)$ with $\sigma = 1$, $\sigma = 0.8$ and $\sigma = 0.67$.

$\mathscr{B}(K_1^D)$			
$(1578,\ \beta_3\beta_4)$	$(56,\ \beta_3\beta_5)$	$(12,\ \beta_1\beta_4)$	$(3456,\ \beta_5)$
$(12346,\ \beta_1)$	$(16,\ \beta_1\beta_3)$	$(1,\ \beta_1\beta_3\beta_4)$	$(346,\ \beta_1\beta_2\beta_5)$
$(2346,\ \beta_1\beta_2)$	$(6,\ \beta_1\beta_2\beta_3\beta_5)$	$(2,\ \beta_1\beta_2\beta_4)$	$(5,\ \beta_3\beta_4\beta_5)$
$(15678,\ \beta_3)$	$(12578,\ \beta_4)$	$(G,\ \emptyset)$	$(\emptyset,\ D_1)$
$\mathscr{B}(K_{0.8}^D)$			
$(1578,\ \beta_3\beta_4)$	$(56,\ \beta_3\beta_5)$	$(12,\ \beta_4\beta_6)$	$(3456,\ \beta_5)$
$(12346,\ \beta_6)$	$(16,\ \beta_3\beta_6)$	$(1,\ \beta_3\beta_4\beta_6)$	$(346,\ \beta_5\beta_6)$
$(15678,\ \beta_3)$	$(6,\ \beta_3\beta_5\beta_6)$	$(G,\ \emptyset)$	$(5,\ \beta_3\beta_4\beta_5)$
$(12578,\ \beta_4)$	$(\emptyset,\ D_{0.8})$		
$\mathscr{B}(K_{0.67}^D)$			
$(12346,\ \beta_6)$	$(56,\ \beta_5\beta_7)$	$(G,\ \emptyset)$	$(3456,\ \beta_5)$
$(125678,\ \beta_3\beta_4)$	$(126,\ \beta_6\beta_7)$	$(6,\ \beta_5\beta_6\beta_7)$	$(346,\ \beta_5\beta_6)$

inference rule 1. Hence, $\alpha_4\alpha_5\alpha_6\alpha_7 \xrightarrow{((1,1)} \beta_3\beta_4$ can be inferred from the maximal (1, 1)-concept rule $\alpha_2\alpha_6 \xrightarrow{(1,1)} \beta_3\beta_4$ by the (1, 1)-decision inference.

Our work can be viewed as an extension and improvement of Ref. [22] to some extent. We illustrate the comparison with an example. For the decision context in Table 15, the decision rules generated by the algorithm in [22] are listed in Table 16 and the decision rules in the present study are listed in Table 17. Obviously, the number of decision rules is much less in Table 17 than in Table 16. Moreover, the decision rules in [22] are essentially (1,1)-rules; in other words, the decision rule in [22] is a special case of the (θ, σ)-rule in this paper.

6. Conclusions

This paper introduced GrC into FCA, and proposed FCA based on fuzzy granularity base for different granulations. The fuzzy granularity base not only offers a uniform technology for knowledge acquisition but also facilitates knowledge-sharing. Using a fuzzy granularity base, we investigated δ-concept lattices, δ-rule extraction and (θ, σ)-rule extraction. Although many inference rules can be used to eliminate superfluous (θ, σ)-rules, the (θ, σ)-inference rules proposed here are perhaps more intuitive and simpler compared to other inference rules. Our results confirm the feasibility of introducing GrC into FCA theoretically and can be regarded as an initial exploration of GrC and FCA combination. We can conclude that our proposed method is probably a much better choice in some FCA applications. In future work we will study application of this theory to multi-value formal contexts and incomplete formal contexts, and will explore wider combination of FCA and GrC.

Table 13
$\Delta_{(\theta,\sigma)}$ with $(\theta,\sigma)=(1,\ 1)$, $(\theta,\sigma)=(1,\ 0.8)$ and $(\theta,\sigma)=(1,\ 0.67)$.

$\Delta_{(1,\ 1)}$	
$\alpha_8 \xrightarrow{(1,\ 1)} \beta_4$	$\emptyset \xrightarrow{(1,\ 1)} \emptyset$
$\alpha_3\alpha_6\alpha_7 \xrightarrow{(1,\ 1)} \beta_5$	$\alpha_5 \xrightarrow{(1,\ 1)} \beta_3$
$\alpha_1\alpha_3\alpha_6\alpha_7 \xrightarrow{(1,\ 1)} \beta_1\beta_2\beta_5$	$\alpha_2\alpha_6 \xrightarrow{(1,\ 1)} \beta_3\beta_4$
$\alpha_1\alpha_2\alpha_5\alpha_6\alpha_7\alpha_8 \xrightarrow{(1,\ 1)} \beta_1\beta_3\beta_4$	$\alpha_3\alpha_5 \xrightarrow{(1,\ 1)} \beta_3\beta_5$
$\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8 \xrightarrow{(1,\ 1)} \beta_3\beta_4\beta_5$	
$\Delta_{(1,\ 0.8)}$	
$\alpha_8 \xrightarrow{(1,\ 0.8)} \beta_4$	$\emptyset \xrightarrow{(1,\ 0.8)} \emptyset$
$\alpha_3\alpha_6\alpha_7 \xrightarrow{(1\ 0.8)} \beta_5$	$\alpha_5 \xrightarrow{(1,\ 0.8)} \beta_3$
$\alpha_1\alpha_3\alpha_6\alpha_7 \xrightarrow{(1\ 0.8)} \beta_5\beta_6$	$\alpha_2\alpha_6 \xrightarrow{(1,\ 0.8)} \beta_3\beta_4$
$\alpha_1\alpha_2\alpha_5\alpha_6\alpha_7\alpha_8 \xrightarrow{(1\ 0.8)} \beta_3\beta_4\beta_6$	$\alpha_3\alpha_5 \xrightarrow{(1,\ 0.8)} \beta_3\beta_5$
$\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8 \xrightarrow{(1\ 0.8)} \beta_3\beta_4\beta_5$	
$\Delta_{(1,\ 0.67)}$	
$\alpha_8 \xrightarrow{(1,\ 0.67)} \beta_7$	$\emptyset \xrightarrow{(1,\ 0.67)} \emptyset$
$\alpha_3\alpha_6\alpha_7 \xrightarrow{(1,\ 0.67)} \beta_5$	$\alpha_5 \xrightarrow{(1,\ 0.67)} \beta_7$
$\alpha_1\alpha_3\alpha_6\alpha_7 \xrightarrow{(1,\ 0.67)} \beta_5\beta_6$	$\alpha_2\alpha_6 \xrightarrow{(1,\ 0.67)} \beta_7$
$\alpha_1\alpha_2\alpha_5\alpha_6\alpha_7\alpha_8 \xrightarrow{(1,\ 0.67)} \beta_6\beta_7$	$\alpha_3\alpha_5 \xrightarrow{(1,\ 0.67)} \beta_5\beta_7$

Table 14
$\Delta_{(\theta,\sigma)}$ with $(\theta,\sigma)=(0.67,\ 1)$, $(\theta,\sigma)=(0.67,\ 0.8)$ and $(\theta,\sigma)=(0.67,\ 0.67)$.

$\Delta_{(0.67,\ 1)}$	
$\alpha_5\alpha_9 \xrightarrow{(0.67,\ 1)} \beta_3\beta_4$	$\emptyset \xrightarrow{(0.67,\ 1)} \emptyset$
$\alpha_4\alpha_9 \xrightarrow{(0.67,\ 1)} \beta_3\beta_4$	$\alpha_5 \xrightarrow{(0.67,\ 1)} \beta_3$
$\alpha_8\alpha_9 \xrightarrow{(0.67,\ 1)} \beta_3\beta_4$	$\alpha_8 \xrightarrow{(0.67,\ 1)} \beta_4$
$\alpha_3\alpha_4\alpha_5\alpha_8\alpha_9 \xrightarrow{(1,\ 1)} \beta_3\beta_4\beta_5$	$\alpha_3\alpha_5 \xrightarrow{(0.67,\ 1)} \beta_3\beta_5$
$\Delta_{(0.67,\ 0.8)}$	
$\alpha_5\alpha_9 \xrightarrow{(0.67,\ 0.8)} \beta_3\beta_4$	$\emptyset \xrightarrow{(0.67,\ 0.8)} \emptyset$
$\alpha_4\alpha_9 \xrightarrow{(0.67,\ 0.8)} \beta_3\beta_4$	$\alpha_5 \xrightarrow{(0.67,\ 0.8)} \beta_3$
$\alpha_8\alpha_9 \xrightarrow{(0.67,\ 0.8)} \beta_3\beta_4$	$\alpha_8 \xrightarrow{(0.67,\ 0.8)} \beta_4$
$\alpha_3\alpha_4\alpha_5\alpha_8\alpha_9 \xrightarrow{(0.67,\ 0.8)} \beta_3\beta_4\beta_5$	$\alpha_3\alpha_5 \xrightarrow{(0.67,\ 0.8)} \beta_3\beta_5$
$\Delta_{(0.67,\ 0.67)}$	
$\alpha_8 \xrightarrow{(0.67,\ 0.67)} \beta_7$	$\emptyset \xrightarrow{(0.67,\ 0.67)} \emptyset$
$\alpha_4\alpha_9 \xrightarrow{(0.67,\ 0.67)} \beta_7$	$\alpha_5 \xrightarrow{(0.67,\ 0.67)} \beta_7$
$\alpha_3\alpha_5 \xrightarrow{(0.67,\ 0.67)} \beta_5\beta_7$	

Table 15
A sample data set [22].

G	a_1	a_2	a_3	a_4	a_5	a_6	d_1	d_2
1	×	×			×	×	×	×
2			×					×
3	×		×			×	×	
4		×						
5		×	×	×	×	×	×	×
6			×		×			
7		×	×	×		×		×
8	×	×		×	×	×	×	

Table 16
The set of decision rules generated by the algorithm in [22].

$\emptyset \to \emptyset$	$a_4a_5 \to d_1$	$a_1a_3a_4 \to d_1d_2$	$a_3a_4a_5 \to d_1d_2$
$a_1 \to d_1$	$a_5a_6 \to d_1$	$a_1a_3a_5 \to d_1d_2$	$a_3a_5a_6 \to d_1d_2$
$a_2a_3 \to d_2$	$a_1a_2a_3 \to d_1d_2$	$a_2a_3a_5 \to d_1d_2$	

Table 17
The set of decision rules $\Delta_{(1,1)}$).

$\emptyset \to \emptyset$	$a_1a_6 \to d_1$	$a_2a_5a_6 \to d_1$
$a_2a_3a_4a_6 \to d_1$	$a_2a_3a_4a_5a_6 \to d_1d_2$	

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