Multigranulation decision-theoretic rough sets

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ABSTRACT

The Bayesian decision-theoretic rough sets propose a framework for studying rough set approximations using probabilistic theory, which can interpret the parameters from existing forms of probabilistic approaches to rough sets. Exploring rough sets in the viewpoint of multigranulation is becoming one of desirable directions in rough set theory, in which lower/upper approximations are approximated by granular structures induced by multiple binary relations. Through combining these two ideas, the objective of this study is to develop a new multigranulation rough set model, called a multigranulation decision-theoretic rough set. Many existing multigranulation rough set models can be derived from the multigranulation decision-theoretic rough set framework.

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1. Introduction

Rough set theory, originated by Pawlak [24,25], has become a well-established theory for uncertainty management in a wide variety of applications related to pattern recognition, image processing, feature selection, neural computing, conflict analysis, decision support, data mining and knowledge discovery [3,5,10,11,15,16,28–31,34,36,41,55]. In the past ten years, several extensions of the rough set model have been proposed in terms of various requirements, such as the decision-theoretic rough set model (see [51]), the variable precision rough set (VPRS) model (see [56,58]), the rough set model based on tolerance relation (see [12–14]), the Bayesian rough set model (see [37]), the Dominance-based rough set model (see [4]), game-theoretic rough set model (see [6,7]), the fuzzy rough set model and the rough fuzzy set model (see [2]).

Recently, the probabilistic rough sets have been paid close attentions [8,45,48,50,52]. A special issue on probabilistic rough sets was set up in International Journal of Approximate Reasoning, in which six relative papers were published [48]. Yao presented a new decision making method based on the decision-theoretic rough set, which is constructed by positive region, boundary region and negative region, respectively [52]. In the literature [50], the author further emphasized the superiority of three-way decisions in probabilistic rough set models. In fact, the probabilistic rough sets are developed based on the Bayesian decision principle, in which its parameters can be learned from a given decision table. Three-way decisions are most of superiorities of probabilistic rough set models. The decision-theoretic rough sets can derive various existing rough set models through setting the thresholds α and β. Since the decision-theoretic rough sets was proposed by Yao [49], it have attracted more and more concerns. Azam and Yao [1] proposed a threshold configuration mechanism for reducing the overall uncertainty of probabilistic regions in the probabilistic rough sets. Jia et al. [9] developed an optimization representation of decision-theoretic rough set model, and gave a heuristic approach and a particle swarm optimization approach for searching an attribute reduction with a minimum cost. Liu et al. [23] combined the logistic regression and the decision-theoretic rough set into a new classification approach, which can effectively reduce the misclassification rate. Yu et al. [53] applied decision-theoretic rough set model for automatically determining the number of clusters with much smaller time cost.

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In the view of granular computing (proposed by Zadeh [54]), in existing rough set models, a general concept described by a set is always characterized via the so-called upper and lower approximations under a single granulation, i.e., the concept is depicted by known knowledge induced from a single relation (such as equivalence relation, tolerance relation and reflexive relation) on the universe [17,18,51]. Conveniently, this kind of rough set models is called single granulation rough sets, just SGRS. In many circumstances, we often need to describe concurrently a target concept through multi binary relations according to a user's requirements or targets of problem solving. Based on this consideration, Qian et al. [26–28] introduced multigranulation rough set theory (MGRS) to more widely apply rough set theory in practical applications, in which lower/upper approximations are approximated by granular structures induced by multi binary relations. From the viewpoint of rough set's applications, the multigranulation rough set theory is very desirable in many real applications, such as multi-source data analysis, knowledge discovery from data with high dimensions and distributive information systems.

Since the multigranulation rough set was proposed by Qian in 2006 [26], the theoretical framework have been largely enriched, and many extended multigranulation rough set models and relative properties and applications have also been proposed and studied [27–32]. Wu and Leung [39] proposed a formal approach to granular computing with multi-scale data measured at different levels of granulations, and studied theory and applications of granular labelled partitions in multi-scale decision information systems. Tripathy et al. [38] developed an incomplete multigranulation rough sets in the context of intuitionistic fuzzy rough sets and gave some important properties of the new rough set model. Raghavan and Tripathy [33] first researched the topological properties of multigranulation rough sets. Based on the idea of multigranulation rough sets, Xu et al. [42–44] developed a variable multigranulation rough set model, a fuzzy multigranulation rough set model and an ordered multigranulation rough set model. Wu [40] extended classical multigranualtion rough sets to the version based on a fuzzy relation, and proposed a new multigranulation fuzzy rough set (MGFRS). Zhang et al. [57] defined a variable precision multigranulation rough set, in which the optimistic multigranulation rough sets and the pessimistic one can be regarded as two extreme cases. Through introducing some membership parameters, this model becomes a multigranulation rough set with dynamic adaption according to practical acquirements. Yang et al. [46,47] examined the fuzzy multigranulation rough set theory, and revealed the hierarchical structure properties of the multigranulation rough sets. Liu and Miao [21,22] established a multigranulation rough set approach in covering contexts. Liang et al. [19] presented a kind of efficient feature selection algorithms for large scale data with a multigranulation strategy. She et al. [35] explored the topological structures and the properties of multigranulation rough sets. Lin et al. [20] gave a neighborhood multigranulation rough set model for multigranulation rough data analysis in the context of hybrid data. In the murigranulation rough set theory, each of various binary relation determines a corresponding information granulation, which largely impacts the commonality between each of the granulations and the fusion among all granulations. As one of very important rough set models, the decision-theoretic rough sets (DTRS) are still not be researched in the context of multigranulation, which limits its further applications in many problems, such as multi-source data analysis, knowledge discovery from data with high dimensions and distributive information systems.

In what follows, besides those motivations mentioned in first multigranulation rough set paper (see Cases 1–3 in the literature [29]), we further emphasize the specific interest of multigranulation rough sets, which can be illustrated from the following three aspects.

- Multigranulation rough set theory is a kind of of information fusion strategies through single granulation rough sets. Optimistic version and pessimistic version are only two simple methods in these information fusion approaches, which are used to easily introduce multigranulation ideas to rough set theory.
- In fact, there are some other fusion strategies [20,39–41,43]. For instance, in the literature [39], Xu et al. introduced a supporting characteristic function and a variable precision parameter β, called an information level, to investigate a target concept under majority granulations.
- With regard to some special information systems, such as multi-source information systems, distributive information systems and groups of intelligent agents, the classical rough sets can not be used to data mining from these information systems, but multigranulation rough sets can.

In this study, our objective is to develop a new multigranulation rough decision theory through combining the multigranulation idea and the Bayesian decisoin theory, called multigranulation decision-theoretic rough sets (MG-DTRS). We mainly give three common forms, the mean multigranulation decision-theoretic rough sets, the optimistic multigranulation decision-theoretic rough sets, and the pessimistic multigranulation decision-theoretic rough sets.

The study is organized as follows. Some basic concepts in classical rough sets and multigranulation rough sets are briefly reviewed in Section 2. In Section 3, we first analyze the loss function and the entire decision risk in the context of multigranulation. Then, we propose three multigranulation decision-theoretic rough set forms that include the mean multigranulation decision-theoretic rough sets, the optimistic multigranulation decision-theoretic rough sets, and the pessimistic multigranulation decision-theoretic rough sets. When the thresholds have a special constraint, the multigranulation decision-theoretic rough sets will produce one of various variables of multigranulation rough sets. In Section 4, we establish the relationships among multigranulation decision-theoretic rough sets (MG-DTRS), other MGRS models, single granulation decision-theoretic rough sets (SG-DTRS) and other SGRS models. Finally, Section 5 concludes this paper by bringing some remarks and discussions.

2. Preliminary knowledge on rough sets

In this section, we review some basic concepts such as information system, Pawlak's rough set, and optimistic multigranulation rough set. Throughout this paper, we assume that the universe U is a finite non-empty set.

2.1. Pawlak's rough set

Formally, an information system can be considered as a pair $I = \langle U, AT \rangle$, where

- U is a non-empty finite set of objects, it is called the universe;
- AT is a non-empty finite set of attributes, such that $\forall a \in AT$, V_a is the domain of attribute a.

$\forall x \in U$, we denote the value of x under the attribute a $(a \in AT)$ by $a(x)$. Given $A \subseteq AT$, an indiscernibility relation $IND(A)$ can be defined as

$$IND(A) = \{(x, y) \in U \times U : a(x) = a(y), \ a \in A\}. \tag{1}$$

The relation $IND(A)$ is reflexive, symmetric and transitive, then $IND(A)$ is an equivalence relation. By the indiscernibility relation $IND(A)$, one can derive the lower and upper approximations of an arbitrary subset X of U. They are defined as

$$\underline{A}(X) = \{x \in U : [x]_A \subseteq X\} \text{ and } \overline{A}(X) = \{x \in U : [x]_A \cap X \neq \emptyset\} \tag{2}$$

respectively, where $[x]_A = \{y \in U : (x, y) \in IND(A)\}$ is the A–equivalence class containing x. The pair $[\underline{A}(X), \overline{A}(X)]$ is referred to as the Pawlak's rough set of X with respect to the set of attributes A.

2.2. Multigranulation rough sets

The multigranulation rough set (MGRS) is different from Pawlak's rough set model because the former is constructed on the basis of a family of indiscernibility relations instead of single indiscernibility relation.

In optimistic multigranulation rough set approach, the word "optimistic" is used to express the idea that in multi independent granular structures, we need only at least one granular structure to satisfy with the inclusion condition between equivalence class and the approximated target. The upper approximation of optimistic multigranulation rough set is defined by the complement of the lower approximation.

Definition 1 [32]. Let I be an information system in which $A_1, A_2, \ldots, A_m \subseteq AT$, then $\forall X \subseteq U$, the optimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{i=1}^{m} A_i}^{O}(X)$ and $\overline{\sum_{i=1}^{m} A_i}^{O}(X)$, respectively,

$$\underline{\sum_{i=1}^{m} A_i}^{O}(X) = \{x \in U : [x]_{A_1} \subseteq X \vee [x]_{A_2} \subseteq X \vee \cdots \vee [x]_{A_m} \subseteq X\}; \tag{3}$$

$$\overline{\sum_{i=1}^{m} A_i}^{O}(X) = \sim\left(\underline{\sum_{i=1}^{m} A_i}^{O}(\sim X)\right); \tag{4}$$

where $[x]_{A_i}$ $(1 \leq i \leq m)$ is the equivalence class of x in terms of set of attributes A_i, and $\sim X$ is the complement of X.

By the lower approximation $\underline{\sum_{i=1}^{m} A_i}^{O}(X)$ and upper approximation $\overline{\sum_{i=1}^{m} A_i}^{O}(X)$, the optimistic multigranulation boundary region of X is

$$BN^{O}_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^{O}(X) - \underline{\sum_{i=1}^{m} A_i}^{O}(X). \tag{5}$$

Proposition 1. Let I be an information system in which $A_1, A_2, \ldots, A_m \subseteq AT$, then $\forall X \subseteq U$, we have

$$\overline{\sum_{i=1}^{m} A_i}^{O}(X) = \{x \in U : [x]_{A_1} \cap X \neq \emptyset \wedge [x]_{A_2} \cap X \neq \emptyset \wedge \cdots \wedge [x]_{A_m} \cap X \neq \emptyset\}. \tag{6}$$

Proof. By Definition 1, we have

$$x \in \overline{\sum_{i=1}^{m} A_i}^{O}(X) \Leftrightarrow x \notin \underline{\sum_{i=1}^{m} A_i}^{O}(\sim X)$$

$$\Leftrightarrow [x]_{A_1} \not\subseteq (\sim X) \wedge [x]_{A_2} \not\subseteq (\sim X) \wedge \cdots \wedge [x]_{A_m} \not\subseteq (\sim X)$$

$$\Leftrightarrow [x]_{A_1} \cap X \neq \emptyset \wedge [x]_{A_2} \cap X \neq \emptyset \wedge \cdots \wedge [x]_{A_m} \cap X \neq \emptyset. \quad \square$$

From Proposition 1, it can be seen that though the optimistic multigranulation upper approximation is defined by the complement of the optimistic multigranulation lower approximation, it can also be considered as a set in which objects have non–empty intersection with the target in terms of each granular structure.

Based on the SCED strategy, the following definition gives the formal representation of lower/upper approximation in the context of multi granular structures.

Definition 2. Let I be an information system in which $A_1, A_2, \ldots, A_m \subseteq AT$, then $\forall X \subseteq U$, the pessimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{i=1}^{m} A_i}^{P}(X)$ and $\overline{\sum_{i=1}^{m} A_i}^{P}(X)$, respectively,

$$\underline{\sum_{i=1}^{m} A_i}^{P}(X) = \{x \in U : [x]_{A_1} \subseteq X \wedge [x]_{A_2} \subseteq X \wedge \cdots \wedge [x]_{A_m} \subseteq X\}; \tag{7}$$

$$\overline{\sum_{i=1}^{m} A_i}^{P}(X) = \sim \left(\underline{\sum_{i=1}^{m} A_i}^{P}(\sim X) \right). \tag{8}$$

By the lower approximation $\underline{\sum_{i=1}^{m} A_i}^{P}(X)$ and upper approximation $\overline{\sum_{i=1}^{m} A_i}^{P}(X)$, the pessimistic multigranulation boundary region of X is

$$BN^{P}_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^{P}(X) - \underline{\sum_{i=1}^{m} A_i}^{P}(X). \tag{9}$$

Proposition 2. Let I be an information system in which $A_1, A_2, \ldots, A_m \subseteq AT$, then $\forall X \subseteq U$, we have

$$\overline{\sum_{i=1}^{m} A_i}^{P}(X) = \{x \in U : [x]_{A_1} \cap X \neq \emptyset \vee [x]_{A_2} \cap X \neq \emptyset \vee \cdots \vee [x]_{A_m} \cap X \neq \emptyset\}. \tag{10}$$

Proof. By Definition 2, we have

$$x \in \overline{\sum_{i=1}^{m} A_i}^{P}(X) \Leftrightarrow x \notin \underline{\sum_{i=1}^{m} A_i}^{P}(\sim X)$$

$$\Leftrightarrow [x]_{A_1} \not\subseteq (\sim X) \vee [x]_{A_2} \not\subseteq (\sim X) \vee \cdots \vee [x]_{A_m} \not\subseteq (\sim X)$$

$$\Leftrightarrow [x]_{A_1} \cap X \neq \emptyset \vee [x]_{A_2} \cap X \neq \emptyset \vee \cdots \vee [x]_{A_m} \cap X \neq \emptyset. \quad \square$$

Different from the upper approximation of optimistic multigranulation rough set, the upper approximation of pessimistic multigranulation rough set is represented as a set in which objects have non–empty intersection with the target in terms of at least one granular structure.

3. MG-DTRS: multigranulation decision-theoretic rough sets

Probabilistic approaches to rough sets have many forms, such as the decision-theoretic rough set model (DTRS) [49,52,50], the variable precision rough set model [58], the Bayesian rough set model [37], and other related studies. Specially, the decision-theoretic rough sets proposed by Yao [49,50,52] has very strong theoretical basis and sound semantic interpretation. Through giving special thresholds, the decision-theoretic rough set model can degenerate into the classical Pawlak rough sets, the variable precision rough set, the 0.5-probabilistic rough set, and so on. In many real applications such as multi-source data analysis, knowledge discovery from data with high dimensions and distributive information systems, if one applies the decision-theoretic rough sets in these cases, the multigranulation version of DTRS will be very desirable. In this section, we will establish a multigranulation decision-theoretic rough set framework.

3.1. Decision-theoretic rough sets

In this subsection, we briefly review some basic concepts in decision-theoretic rough sets.

In the Bayesian decision procedure, a finite set of states can be written as $\Omega = \{\omega_1, \omega_2, \ldots, \omega_s\}$, and a finite set of m possible actions can be denoted by $A = \{a_1, a_2, \cdots, a_r\}$. Let $P(\omega_j|\mathbf{x})$ be the conditional probability of an object x being in state ω_j given that the object is described by $\mathbf{x}$. Let $\lambda(a_i|\omega_j)$ denote the loss, or cost, for taking action a_i when the state is ω_j, the expected loss associated with taking action a_i is given by

$$R(a_i|\mathbf{x}) = \sum_{j=1}^{s} \lambda(a_i|\omega_j)P(\omega_j|\mathbf{x}).$$

In classical rough set theory, the approximation operators partition the universe into three disjoint classes $POS(A)$, $NEG(A)$, and $BND(A)$. Through using the conditional probability $P(X|[x])$, the Bayesian decision precedure can decide how to assign x into these three disjoint regions [50,52]. The expected loss $R(a_i|[x])$ associated with taking the individual actions can be expressed as

$$R(a_1|[x]) = \lambda_{11}P(X|[x]) + \lambda_{12}P(X^c|[x]),$$
$$R(a_2|[x]) = \lambda_{21}P(X|[x]) + \lambda_{22}P(X^c|[x]),$$
$$R(a_3|[x]) = \lambda_{31}P(X|[x]) + \lambda_{32}P(X^c|[x]),$$

where $\lambda_{i1} = \lambda(a_i|X)$, $\lambda_{i2} = \lambda(a_i|X^c)$, and $i = 1, 2, 3$. When $\lambda_{11} \leq \lambda_{31} < \lambda_{21}$ and $\lambda_{22} \leq \lambda_{32} < \lambda_{12}$, the Bayesian decision procedure leads to the following minimum-risk decision rules:

(P) If $P(X|[x]) \geq \gamma$ and $P(X|[x]) \geq \alpha$, decision $POS(X)$;
(N) If $P(X|[x]) \leq \beta$ and $P(X|[x]) \leq \gamma$, decision $NEG(X)$;
(B) If $\beta \leq P(X|[x]) \leq \alpha$, decide $BND(X)$;

where

$$\alpha = \frac{\lambda_{12} - \lambda_{32}}{(\lambda_{31} - \lambda_{32}) - (\lambda_{11} - \lambda_{12})}, \quad \gamma = \frac{\lambda_{12} - \lambda_{22}}{(\lambda_{21} - \lambda_{22}) - (\lambda_{11} - \lambda_{12})}, \quad \beta = \frac{\lambda_{32} - \lambda_{22}}{(\lambda_{21} - \lambda_{22}) - (\lambda_{31} - \lambda_{32})}.$$

If a loss function with $\lambda_{11} \leq \lambda_{31} < \lambda_{21}$ and $\lambda_{22} \leq \lambda_{32} < \lambda_{12}$ further satisfies the condition:
$(\lambda_{12} - \lambda_{32})(\lambda_{21} - \lambda_{31}) \geq (\lambda_{31} - \lambda_{11})(\lambda_{32} - \lambda_{22})$,
then $\alpha \geq \gamma \geq \beta$.
When $\alpha > \beta$, we have $\alpha > \gamma > \beta$. The decision-theoretic rough set has the decision rules:

(P1) If $P(X|[x]) \geq \alpha$, decide $POS(X)$;
(N1) If $P(X|[x]) \leq \beta$, decide $NEG(X)$;
(B1) If $\beta < P(X|[x]) < \alpha$, decide $BND(X)$.

Using these three decision rules, we get the probabilistic approximation:
$\underline{apr}_\alpha(X) = \{x : P(X|[x]) \geq \alpha, \ x \in U\}$,
$\overline{apr}_\beta(X) = \{x : P(X|[x]) > \beta, \ x \in U\}$.
When $\alpha = \beta$, we have $\alpha = \gamma = \beta$. The decision-theoretic rough set has the following decision rules:

(P2) If $P(X|[x]) > \alpha$, decide $POS(X)$;
(N2) If $P(X|[x]) < \alpha$, decide $NEG(X)$;
(B2) If $P(X|[x]) = \alpha$, decide $BND(X)$.

Using the above three decision rules, we get the probabilistic approximation:

$$\underline{apr}_\alpha(X) = \{x : P(X|[x]) > \alpha, \ x \in U\},$$
$$\overline{apr}_\alpha(X) = \{x : P(X|[x]) \geq \alpha, \ x \in U\}.$$

In the framework of decision-theoretic rough sets, the Pawlak rough set model, the variable precision rough set model, the Bayesian rough set model and the 0.5-probabilisitc rough set model can be pooled together and studied based on the notions of conditional functions.

3.2. Theoretical foundation in multigranulation decision-theoretic rough sets

The multigranulation rough set (MGRS) is different from Pawlak's rough set model because the former is constructed on the basis of a family of indiscernibility relations instead of single indiscernibility relation.

Given $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$ m granular structures and $\forall X \subseteq U$, the lower/upper approximation in a multigranulation rough set can be formally represented as two fusion functions, respectively,

$$\sum_{i=1}^{m} \underline{R_i}(X) = f_l(R_1, R_2, \ldots, R_m),$$

$$\sum_{i=1}^{m} \overline{R_i}(X) = f_u(R_1, R_2, \ldots, R_m),$$

where f_l is called a lower fusion function, and f_u is called an upper fusion function. These two functions are used to compute the lower/upper approximation of a multigranulation rough set through fusing m granular structures.

In practical applications of multigranulation rough sets, the fusion function has many forms according to various semantics and requirements. Conveniently, let $\lambda_k(a_i|\omega_j)$ denote the loss, or cost, for taking action a_i when the state is ω_j by k-th granular structures. Let $P(\omega_j|\mathbf{x}_k)$ be the conditional probability of an object x being in state ω_j given that the object is described by $\mathbf{x}_k$ under k-th granular structures. The expected loss associated with taking action a_i is given by

$$R(a_i|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) = \sum_{k=1}^{m} \sum_{j=1}^{s} \lambda_k(a_i|\omega_j)P(\omega_j|\mathbf{x}_k). \tag{11}$$

The expected loss $R(a_i|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ is a conditional risk. $\tau(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ specifies which action to take, and its value is one of the actions $a_1, a_2, \ldots, a_r$. The overall risk $\mathbf{R}$ is the expected loss associated with the decision rule $\tau(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$, the overall risk is defined by

$$\mathbf{R} = \sum_{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m} R(\tau(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)\mathbf{P}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m), \tag{12}$$

where $\mathbf{P}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ is a joint probability, which is calculated through fusing $(P(\mathbf{x}_1), P(\mathbf{x}_2), \ldots, P(\mathbf{x}_m))$ induced by m granular structures induced by the same universe.

Given multiple granular structures $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$, the multigranulation decision-theoretic rough sets aim to select a series of actions for which the overall risk is as small as possible, in which the actions include deciding positive region, deciding negative region and deciding boundary region.

In the multigranulation decision-theoretic rough sets, there are two kinds of assumptions. One assumes that the values of $\lambda_k(a_i|\omega_j)$, $k \leq m$, are all equal each other, and the other assumes that they are not equivalent, in which each granular structure has its independent loss (or cost) functions itself. In order to introduce the idea of multigranulation decision-theoretic rough sets, this paper only deals with the first assumption. Hence, the determined procedure of the parameters α, β and γ is consistent with that of classical decision-theoretic rough sets, and the value of each parameter in every granular structure is also equal each other. The multigranulation decision-theoretic rough sets for the second assumption will be established in future work.

3.3. Three cases of multigranulation decision-theoretic rough sets

Given m granular structures $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$, when $\lambda_k(a_i|\omega_j) = \lambda_l(a_i|\omega_j)$, $k, l \in \{1, 2, \ldots, m\}$, the expected loss associated with taking action a_i can be given by

$$R(a_i|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) = \sum_{k=1}^{m} \sum_{j=1}^{s} \lambda(a_i|\omega_j)P(\omega_j|\mathbf{x}_k). \tag{13}$$

In this case, the information fusion in multigranulation decision-theoretic rough sets can be simplified as the fusion of a set of probabilities under the same universe. In this subsection, we give three multigranulation decision-theoretic rough set models, which are a mean multigranulation decision-theoretic rough set (MMG-DTRS), an optimistic multigranulation decision-theoretic rough set (OMG-DTRS) and a pessimistic multigranulation decision-theoretic rough set (PMG-DTRS), respectively.

3.3.1. Mean multigranulation decision-theoretic rough sets

In multigranulation decision-theoretic rough sets, when the loss function is fixed, judging the conditional probability of an object x within a target concept in m granular structures can be computed by its mathematic expectation. That is to say,

$$E(P(X|x)) = P(X|[x]_{R_1}) + P(X|[x]_{R_2}) + \cdots + P(X|[x]_{R_m}))/m. \tag{14}$$

The joint probability is estimated by the mean value of m conditional probabilities. Based on this idea, hence we give a kind of multigranulation decision-theoretic rough set, called mean multigranulation decision-theoretic rough sets. Its formal definition is as follows.

Definition 3. Given $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$ m granular structures and $\forall X \subseteq U$, the mean multigranulation lower and upper approximations are denoted by $\underline{\sum_{i=1}^{m} R_i}^{M,\,\alpha}(X)$ and $\overline{\sum_{i=1}^{m} R_i}^{M,\,\beta}(X)$, respectively,

$$\underline{\sum_{i=1}^{m} R_i}^{M,\,\alpha}(X) = \{x : (P(X|[x]_{R_1}) + P(X|[x]_{R_2}) + \cdots + P(X|[x]_{R_m}))/m \geq \alpha, x \in U\}; \tag{15}$$

$$\overline{\sum_{i=1}^{m} R_i}^{M,\,\beta}(X) = U - \{x : (P(X|[x]_{R_1}) + P(X|[x]_{R_2}) + \cdots + P(X|[x]_{R_m}))/m \leq \beta, x \in U\}; \tag{16}$$

where $[x]_{R_i}$ $(1 \leq i \leq m)$ is the equivalence class of x induced by R_i, $P(X|[x]_{R_i})$ is the conditional probability of the equivalent class $[x]_{R_i}$ with respect to X, and α, β are two probability constraints.

By the lower approximation $\underline{\sum_{i=1}^{m} R_i}^{M,\,\alpha}(X)$ and upper approximation $\overline{\sum_{i=1}^{m} R_i}^{M,\,\beta}(X)$, the mean multigranulation boundary region of X is

$$BN^{M}_{\sum_{i=1}^{m} R_i}(X) = \overline{\sum_{i=1}^{m} R_i}^{M,\,\beta}(X) - \underline{\sum_{i=1}^{m} R_i}^{M,\,\alpha}(X). \tag{17}$$

Similar to the classical decision-theoretic rough sets, when the thresholds $\alpha > \beta$, we can obtain the decision rules tie-broke:

(MP1) If $(P(X|[x]_{R_1}) + P(X|[x]_{R_2}) + \cdots + P(X|[x]_{R_m}))/m \geq \alpha$, decide $POS(X)$;
(MN1) If $(P(X|[x]_{R_1}) + P(X|[x]_{R_2}) + \cdots + P(X|[x]_{R_m}))/m \leq \beta$, decide $NEG(X)$;
(MB1) If $\beta < (P(X|[x]_{R_1}) + P(X|[x]_{R_2}) + \cdots + P(X|[x]_{R_m}))/m < \alpha$, decide $BND(X)$.

When $\alpha = \beta$, we have $\alpha = \gamma = \beta$. The mean multigranulation decision-theoretic rough set has the following decision rules:

(MP2) If $(P(X|[x]_{R_1}) + P(X|[x]_{R_2}) + \cdots + P(X|[x]_{R_m}))/m > \alpha$, decide $POS(X)$;
(MN2) If $(P(X|[x]_{R_1}) + P(X|[x]_{R_2}) + \cdots + P(X|[x]_{R_m}))/m < \alpha$, decide $NEG(X)$;
(MB2) If $(P(X|[x]_{R_1}) + P(X|[x]_{R_2}) + \cdots + P(X|[x]_{R_m}))/m = \alpha$, decide $BND(X)$.

3.3.2. Optimistic multigranulation decision-theoretic rough sets

In existing optimistic multigranulation rough set approaches, the word "optimistic" is used to express the idea that in multi independent granular structures, its multigranulation lower approximation only needs at least one granular structure to satisfy with the inclusion condition between an equivalence class and the approximated target. While the upper approximation of an optimistic multigranulation rough set is defined by the complement of the lower approximation. Based on this idea, in this part, we develop an optimistic multigranulation decision-theoretic rough set.

In this optimistic multigranulation decision-theoretic rough set, its lower approximation collects those objects in which each object has at least one granular structure satisfying the probability constraint $(\geq \alpha)$ between its equivalence class and the approximate target, while its upper approximation collects those objects in which each object has all granular structures satisfying the probability constraint $(\leq \beta)$ between its equivalence class and the approximate target.

Definition 4. Given $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$ m granular structures and $\forall X \subseteq U$, the optimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X)$ and $\overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X)$, respectively,

$$\underline{\sum_{i=1}^{m} R_i}^{O,\alpha}(X) = \{x : P(X|[x]_{R_1}) \geq \alpha \vee P(X|[x]_{R_2}) \geq \alpha \vee \cdots \vee P(X|[x]_{R_m}) \geq \alpha, x \in U\}; \tag{18}$$

$$\overline{\sum_{i=1}^{m} R_i}^{O,\beta}(X) = U - \{x : P(X|[x]_{R_1}) \leq \beta \wedge P(X|[x]_{R_2}) \leq \beta \wedge \cdots \vee P(X|[x]_{R_m}) \leq \beta, x \in U\}; \tag{19}$$

where $[x]_{R_i}$ $(1 \leq i \leq m)$ is the equivalence class of x induced by R_i, $P(X|[x]_{R_i})$ is the conditional probability of the equivalent class $[x]_{R_i}$ with respect to X, and α, β are two probability constraints.

By the lower approximation $\underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X)$ and upper approximation $\overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X)$, the optimistic multigranulation boundary region of X is

$$BN^{O}_{\sum_{i=1}^{m} R_i}(X) = \overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X) - \underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X). \tag{20}$$

From the definition of optimistic multigranulation decision-theoretic rough sets, one can obtain the following three propositions.

Proposition 3. *Given $R_1, R_2, \ldots, R_m \subseteq \boldsymbol{R}$ m granular structures and $\forall X \subseteq U$. Then, the following properties hold*

(1) $\underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X) \supseteq \underline{R_{i_\alpha}}(X), i \leq m$;

(2) $\overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X) \subseteq \overline{R_{i\beta}}(X), i \leq m$;

where $\underline{R_{i_\alpha}}(X) = \{x : P(X|[x]_{R_i}) \geq \alpha, \ x \in U\}$, and $\overline{R_{i\beta}}(X) = \{x : P(X|[x]_{R_i}) \geq \beta, \ x \in U\}$.

Proposition 4. *Given $R_1, R_2, \ldots, R_m \subseteq \boldsymbol{R}$ m granular structures and $\forall X \subseteq U$. Then, the following properties hold*

(1) $\underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X) = \bigcup_{i=1}^{m} \underline{R_{i_\alpha}}(X)$;

(2) $\overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X) = \bigcap_{i=1}^{m} \overline{R_{i\beta}}(X)$;

where $\underline{R_{i_\alpha}}(X) = \{x : P(X|[x]_{R_i}) \geq \alpha, \ x \in U\}$, and $\overline{R_{i\beta}}(X) = \{x : P(X|[x]_{R_i}) \geq \beta, \ x \in U\}$.

Proposition 5. *Given $R_1, R_2, \ldots, R_m \subseteq \boldsymbol{R}$ m granular structures and $\forall X_1 \subseteq X_2 \subseteq U$. Then, the following properties hold*

(1) $\underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X_1) \subseteq \underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X_2)$;

(2) $\overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X_1) \subseteq \overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X_2)$.

Similar to the classical decision-theoretic rough sets, when the thresholds $\alpha > \beta$, we can obtain the decision rules tie-broke:

(OP1) If $\exists i \in \{1, 2, \ldots, m\}$ such that $P(X|[x]_{R_i}) \geq \alpha$, decide $POS(X)$;

(ON1) If $\forall i \in \{1, 2, \ldots, m\}$ such that $P(X|[x]_{R_i}) \leq \beta$, decide $NEG(X)$;

(OB1) Otherwise, decide $BND(X)$.

When $\alpha = \beta$, we have $\alpha = \gamma = \beta$. The optimisitc multigranulation decision-theoretic rough set has the following decision rules:

(OP2) If $\exists i \in \{1, 2, \ldots, m\}$ such that $P(X|[x]_{R_i}) > \alpha$, decide $POS(X)$;

(ON2) If $\forall i \in \{1, 2, \ldots, m\}$ such that $P(X|[x]_{R_i}) < \alpha$, decide $NEG(X)$;

(OB2) Otherwise, decide $BND(X)$.

3.3.3. Pessimistic multigranulation decision-theoretic rough sets

In decision making analysis, "Seeking common ground while eliminating differences" (SCED) is one of usual decision strategies. This strategy argues that one reserves common decisions while deleting inconsistent decisions, which can be seen as a conservative decision strategy. Based on this consideration, Qian et al. [31] proposed a so-called pessimistic

multigranulation rough set. In this subsection, we will combine pessimistic multigranulation rough set and decision-theoretic rough set into an entire decision framework together.

In the pessimistic multigranulation decision-theoretic rough sets, its lower approximation collects those objects in which its equivalence class from all granular structures satisfying the probability constraint ($\geq \alpha$) between its equivalence class and the approximate target, while its upper approximation collects those objects in which each object has at least one granular structure satisfying the probability constraint ($\leq \beta$) between its equivalence class and the approximate target.

Definition 5. Given $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$ m granular structures and $\forall X \subseteq U$, the pessimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{i=1}^{m} R_i}^{P, \alpha}(X)$ and $\overline{\sum_{i=1}^{m} R_i}^{P, \beta}(X)$, respectively,

$$\underline{\sum_{i=1}^{m} R_i}^{P, \alpha}(X) = \{x : P(X|[x]_{R_1}) \geq \alpha \wedge P(X|[x]_{R_2}) \geq \alpha \wedge \cdots \wedge P(X|[x]_{R_m}) \geq \alpha, x \in U\}; \tag{21}$$

$$\overline{\sum_{i=1}^{m} R_i}^{P, \beta}(X) = U - \{x : P(X|[x]_{R_1}) \leq \beta \vee P(X|[x]_{R_2}) \leq \beta \vee \cdots \vee P(X|[x]_{R_m}) \leq \beta, x \in U\}; \tag{22}$$

where $[x]_{R_i}$ ($1 \leq i \leq m$) is the equivalence class of x induced by R_i, $P(X|[x]_{R_i})$ is the conditional probability of the equivalent class $[x]_{R_i}$ with respect to X, and α, β are two probability constraints.

By the lower approximation $\underline{\sum_{i=1}^{m} A_i}^{P, \alpha}(X)$ and upper approximation $\overline{\sum_{i=1}^{m} A_i}^{P, \beta}(X)$, the pessimistic multigranulation boundary region of X is

$$BN^{P}_{\sum_{i=1}^{m} R_i}(X) = \overline{\sum_{i=1}^{m} R_i}^{P, \beta}(X) - \underline{\sum_{i=1}^{m} R_i}^{P, \alpha}(X). \tag{23}$$

From the definition of pessimistic multigranulation decision-theoretic rough set, the following three propositions can be easily induced.

Proposition 6. Given $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$ m granular structures and $\forall X \subseteq U$. Then, the following properties hold

(1) $\underline{\sum_{i=1}^{m} R_i}^{P, \alpha}(X) \subseteq \underline{R_{i_\alpha}}(X), i \leq m;$
(2) $\overline{\sum_{i=1}^{m} R_i}^{P, \beta}(X) \supseteq \overline{R_{i\beta}}(X), i \leq m;$

where $\underline{R_{i_\alpha}}(X) = \{x : P(X|[x]_{R_i}) \geq \alpha, x \in U\}$, and $\overline{R_{i\beta}}(X) = \{x : P(X|[x]_{R_i}) \geq \beta, x \in U\}$.

Proposition 7. Given $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$ m granular structures and $\forall X \subseteq U$. Then, the following properties hold

(1) $\underline{\sum_{i=1}^{m} R_i}^{P, \alpha}(X) = \bigcap_{i=1}^{m} \underline{R_{i_\alpha}}(X);$
(2) $\overline{\sum_{i=1}^{m} R_i}^{P, \beta}(X) = \bigcup_{i=1}^{m} \overline{R_{i\beta}}(X);$

where $\underline{R_{i_\alpha}}(X) = \{x : P(X|[x]_{R_i}) \geq \alpha, x \in U\}$, and $\overline{R_{i\beta}}(X) = \{x : P(X|[x]_{R_i}) \geq \beta, x \in U\}$.

Proposition 8. Given $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$ m granular structures and $\forall X_1 \subseteq X_2 \subseteq U$. Then, the following properties hold

(1) $\underline{\sum_{i=1}^{m} R_i}^{P, \alpha}(X_1) \subseteq \underline{\sum_{i=1}^{m} R_i}^{P, \alpha}(X_2);$
(2) $\overline{\sum_{i=1}^{m} R_i}^{P, \beta}(X_1) \subseteq \overline{\sum_{i=1}^{m} R_i}^{P, \beta}(X_2).$

Similar to the classical decision-theoretic rough sets, when the thresholds $\alpha > \beta$, we can obtain the decision rules tie-broke:

(PP1) If $\forall i \in \{1, 2, \ldots, m\}$ such that $P(X|[x]_{R_i}) \geq \alpha$, decide $POS(X)$;
(PN1) If $\exists i \in \{1, 2, \ldots, m\}$ such that $P(X|[x]_{R_i}) \leq \beta$, decide $NEG(X)$;
(PB1) Otherwise, decide $BND(X)$.

When $\alpha = \beta$, we have $\alpha = \gamma = \beta$. The pessimistic multigranulation decision-theoretic rough set has the following decision rules:

(PP2) If $\forall i \in \{1, 2, \ldots, m\}$ such that $P(X|[x]_{R_i}) > \alpha$, decide $POS(X)$;
(PN2) If $\exists i \in \{1, 2, \ldots, m\}$ such that $P(X|[x]_{R_i}) < \alpha$, decide $NEG(X)$;
(PB2) Otherwise, decide $BND(X)$.

The following proposition establishes the relationships among the mean multigranulation decision-theoretic rough sets, the optimistic multigranulation decision-theoretic rough sets, and the pessimistic multigranulation decision-theoretic rough sets.

Proposition 9. *Given $R_1, R_2, \ldots, R_m \subseteq \mathbf{R}$ m granular structures and $\forall X \subseteq U$. Then, the following properties hold*

(1) $\underline{\sum_{i=1}^{m} R_i}^{P,\,\alpha}(X) \subseteq \underline{\sum_{i=1}^{m} R_i}^{M,\,\alpha}(X) \subseteq \underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X)$;
(2) $\overline{\sum_{i=1}^{m} R_i}^{P,\,\beta}(X) \supseteq \overline{\sum_{i=1}^{m} R_i}^{M,\,\beta}(X) \supseteq \overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X)$.

4. Relationships between MG-DTRS and other MGRS models

4.1. Classical multigranulation rough sets

In the decision-theoretic rough sets, the probability value, the thresholds α and β decide its detailed form of rough sets. From Yao's work [50,52], it follows that when $\alpha = 1$ and $\beta = 0$, the decision-theoretic rough sets will degenerate into the standard rough sets. In this case, we have that

$$P(X|[x]_R) = \frac{|[x]_R \cap X|}{|[x]_R|} = 1 \Leftrightarrow [x]_R \subseteq X,$$

$$P(X|[x]_R) = \frac{|[x]_R \cap X|}{|[x]_R|} = 0 \Leftrightarrow [x]_R \cap X = \emptyset.$$

Hence,

$$\underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X) = \{x : P(X|[x]_{R_1}) \geq 1 \vee P(X|[x]_{R_2}) \geq 1 \vee \cdots \vee P(X|[x]_{R_m}) \geq 1, x \in U\}$$

$$\Rightarrow \underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X) = \{x : [x]_{R_1} \subseteq X \vee [x]_{R_2} \subseteq X \vee \cdots \vee [x]_{R_m} \subseteq X, x \in U\},$$

and

$$\overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X) = U - \{x : P(X|[x]_{R_1}) \leq 0 \wedge P(X|[x]_{R_2}) \leq 0 \wedge \cdots \wedge P(X|[x]_{R_m}) \leq 0, x \in U\}$$

$$\Rightarrow \overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X) = \{x : [x]_{R_1} \cap X \neq \emptyset \wedge [x]_{R_2} \cap X \neq \emptyset \wedge \cdots \wedge [x]_{R_m} \cap X \neq \emptyset, x \in U\}.$$

The multigranulation lower approximation and multigranulation upper approximation are consistent with those in classical optimistic multigranulation rough sets (OMGRS) [32]. Hence, when $\alpha = 1$ and $\beta = 0$, the optimistic multigranulation decision-theoretic rough sets (OMG-DTRS) will degenerate into the optimistic multigranulation rough sets (OMGRS). Similarly, one has that

$$\underline{\sum_{i=1}^{m} R_i}^{P,\,\alpha}(X) = \{x : P(X|[x]_{R_1}) \geq 1 \wedge P(X|[x]_{R_2}) \geq 1 \wedge \cdots \wedge P(X|[x]_{R_m}) \geq 1, x \in U\}$$

$$\Rightarrow \underline{\sum_{i=1}^{m} R_i}^{P,\,\alpha}(X) = \{x : [x]_{R_1} \subseteq X \wedge [x]_{R_2} \subseteq X \wedge \cdots \wedge [x]_{R_m} \subseteq X, x \in U\},$$

and

$$\overline{\sum_{i=1}^{m} R_i}^{P,\,\beta}(X) = U - \{x : P(X|[x]_{R_1}) \leq 0 \vee P(X|[x]_{R_2}) \leq 0 \vee \cdots \vee P(X|[x]_{R_m}) \leq 0, x \in U\}$$

$$\Rightarrow \overline{\sum_{i=1}^{m} R_i}^{P,\,\beta}(X) = \{x : [x]_{R_1} \cap X \neq \emptyset \vee [x]_{R_2} \cap X \neq \emptyset \vee \cdots \vee [x]_{R_m} \cap X \neq \emptyset, x \in U\}.$$

The multigranulation lower approximation and multigranulation upper approximation are equivalent to those in classical pessimistic multigranulation rough sets (PMGRS) [31]. Thus, pessimistic multigranulation decision-theoretic rough sets (PMG-DTRS) will degenerate into the pessimistic multigranulation rough sets (PMGRS).

4.2. Variable multigranulation rough sets

When $\alpha + \beta = 1$ and $0 \leq \beta \leq 0.5 < \alpha \leq 1$, the decision-theoretic rough sets become the variable precision rough sets. The condition $0 \leq \beta \leq 0.5 < \alpha \leq 1$ follows that the lower approximation is a subset of the upper approximation. Hence,

$$\underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X) = \{x : P(X|[x]_{R_1}) \geq \alpha \vee P(X|[x]_{R_2}) \geq \alpha \vee \cdots \vee P(X|[x]_{R_m}) \geq \alpha, x \in U\},$$

$$\overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X) = U - \{x : P(X|[x]_{R_1}) \leq \beta \wedge P(X|[x]_{R_2}) \leq \beta \wedge \cdots \wedge P(X|[x]_{R_m}) \leq \beta, x \in U\},$$

$$\Rightarrow \underline{\sum_{i=1}^{m} R_i}^{O,\,\alpha}(X) = \{x : P(X|[x]_{R_1}) \geq \alpha \vee P(X|[x]_{R_2}) \geq \alpha \vee \cdots \vee P(X|[x]_{R_m}) \geq \alpha, x \in U\},$$

$$\overline{\sum_{i=1}^{m} R_i}^{O,\,\beta}(X) = \{x : P(X|[x]_{R_1}) > 1 - \alpha \wedge P(X|[x]_{R_2}) > 1 - \alpha \wedge \cdots \wedge P(X|[x]_{R_m}) > 1 - \alpha, x \in U\}.$$

The multigranulation lower approximation and multigranulation upper approximation are consistent with those in the optimistic variable precision multigranulation rough sets (OVMGRS) proposed by [57]. Hence, when $\alpha + \beta = 1$ and $0 \leq \beta \leq 0.5 < \alpha \leq 1$, OMG-DTRS will degenerate into the optimistic variable precision multigranulation rough sets (OVMGRS).

Similarly, the pessimistic multigranulation decision-theoretic rough sets have the following properties.

$$\underline{\sum_{i=1}^{m} R_i}^{P,\,\alpha}(X) = \{x : P(X|[x]_{R_1}) \geq \alpha \wedge P(X|[x]_{R_2}) \geq \alpha \wedge \cdots \wedge P(X|[x]_{R_m}) \geq \alpha, x \in U\},$$

$$\overline{\sum_{i=1}^{m} R_i}^{P,\,\beta}(X) = U - \{x : P(X|[x]_{R_1}) \leq \beta \vee P(X|[x]_{R_2}) \leq \beta \vee \cdots \vee P(X|[x]_{R_m}) \leq \beta, x \in U\},$$

$$\Rightarrow \underline{\sum_{i=1}^{m} R_i}^{P,\,\alpha}(X) = \{x : P(X|[x]_{R_1}) \geq \alpha \wedge P(X|[x]_{R_2}) \geq \alpha \wedge \cdots \wedge P(X|[x]_{R_m}) \geq \alpha, x \in U\},$$

$$\overline{\sum_{i=1}^{m} R_i}^{P,\,\beta}(X) = \{x : P(X|[x]_{R_1}) > 1 - \alpha \vee P(X|[x]_{R_2}) > 1 - \alpha \vee \cdots \vee P(X|[x]_{R_m}) > 1 - \alpha, x \in U\}.$$

The multigranulation lower approximation and multigranulation upper approximation are equivalent to those in the pessimistic variable precision multigranulation rough sets (PVMGRS) developed by Zhang et al. [57]. Thus, PMG-DTRS will degenerate into the pessimistic variable precision multigranulation rough sets (PVMGRS).

When the thresholds α and β have other constrain relationships, the multigranulation decision-theoretic rough sets will produce various variables of multigranulation rough sets, which can be applied in many practical applications.

Based on the above discussions, we can obtain the relationships among MG-DTRS, other MGRS models, SG-DTRS and other SGRS models, which is shown as Fig. 1. In this figure, MG-PRS means a multigranulation probabilistic rough set, MG-VRS is a multigranulation variable precision rough set, and MG-0.5PRS means a multigranulation 0.5-probabilistic rough set,

(a) Relationship between MG-DTRS and other MGRS models   (b) Relationship between MG-DTRS and SG-DTRS
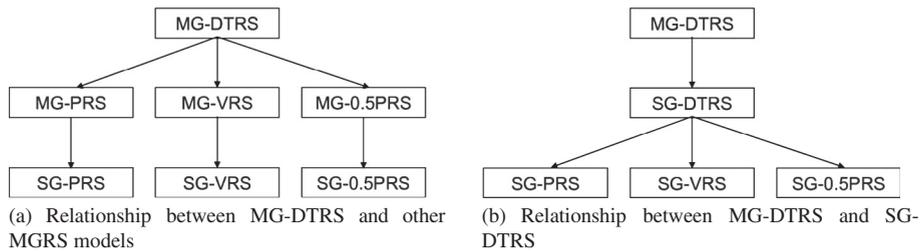
Fig. 1. Relationships among MG-DTRS, other MGRS models, SG-DTRS and other SGRS models.

while SG-PRS means a MG-PRS means a multigranulation probabilistic rough set, probabilistic rough set, SG-VRS is a single granulation variable precision rough set, and MG-0.5PRS means a single granulation 0.5-probabilistic rough set, respectively.

5. Conclusions

Multigranulation rough set theory (MGRS) is one of desirable directions in rough set theory, in which lower/upper approximations are approximated by granular structures induced by multiple binary relations. It provides a new perspective for decision making analysis based on the rough set theory. In this paper, we have first proposed a new multigranulation rough set model through combining MGRS and the decision-theoretic rough sets together, called a multigranulation decision-theoretic rough set model. In this framework, we have given three forms of MG-DTRS, which are mean multigranulation decision-theoretic rough sets, optimistic multigranulation decision-theoretic rough sets, and pessimistic multigranulation decision-theoretic rough sets. These forms of MG-DTRS can derive many existing multigranulation rough set models when the parameters satisfy special constraints. Finally, we have also established the relationships among multigranulation decision-theoretic rough sets, multigranulation rough sets and single granulation rough sets.

This study only develops a framework of multigranulation decision-theoretic rough sets, in which there are still many interesting issues to be explored. Its future direction has four aspects: (1) model extension of multigranulation decision-theoretic rough sets in other types of data sets; (2) information fusion based on multiple granular structures; (3) information granule selection and granulation selection; and (4) applications of multigranulation decision-theoretic rough sets. It is deserved to point out that the multigranulation decision-theoretic rough set and standard decision-theoretic rough set can be combined to data mining and decision making in real applications, such as multi-source information systems, data with high dimensions, distributive information systems.

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