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ORIGINAL ARTICLE

Dependency space, closure system and rough set theory

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Abstract This paper researches on potential relations of dependency space, closure system and rough set theory, and mainly focuses on solving some essential problems of rough set theory based on dependency space and closure system respectively. Firstly, we pretreat an information system into a relatively simple derivative system, in which dependency space and closure system are generated; Secondly, by means of dependency space and closure system separately we can solve some essential problems of rough set theory, such as reducts, cores; Finally, we reveal interior relations between dependency space and closure system. Conclusions of this paper not only help to understand rough set theory from the prospective of the dependency space and closure system, but also provide a new theoretical basis for data analysis and processing.

Keywords Rough set theory · Dependency space · Closure system

1 Introduction

Rough set theory, proposed by Pawlak in the 1980s [10], could effectively find principal or determinant factors from data by reducing attributes, and thus achieve the simplification and refinement of data. At the same time it expands the classical set theory and demonstrates exclusive advantages in dealing with inaccurate and incomplete information [2, 5, 7–9, 15, 16, 21, 22, 25]. In recent years, rough set theory has not only been continuously improved in theoretical sphere, but also has been successfully applied in many practical areas [1, 4, 12–14, 18, 20, 26], such as machine learning, pattern recognition, decision analysis, image processing, medical diagnostics, approximate reasoning, process control, knowledge discovery in databases, expert systems and other fields.

Currently, study of rough set theory is mainly concentrated on its mathematical nature, measured property, and relations with other data analysis tools. In the study of relations between rough set theory and other data analysis tools, there have been many important achievements such as, relations and complementarities between rough set and fuzzy set, Dempster-Shafer evidence [19, 23, 24]. In the task of features extraction from data, the research on relations between neural network and rough set is also an interesting issue [11]. In the reference [17], which aims to establish the relationship between FCA and rough set theory. This paper introduces dependency space and closure system into rough set theory, and elaborates relations of rough set, closure system and dependency space. Conclusions of this paper not only help to understand rough set theory from the prospective of the dependency space and the closure system, but also provide the basis for the further combination of rough set theory, dependency space and closure system.

The paper is structured as follows: in the flowing section, we briefly recall fundamental notions and results involved in dependency space, closure system and rough set theory. Section 3 transforms information system into a

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relatively simple derivative system, in which the dependency space and the closure system are generated. Sections 4 and 5 introduce dependency space and closure system into rough set theory separately. Section 6 reveals relations between dependency space and closure system. Conclusion and discussion of the further work will close the paper in Sect. 7.

2 Basic notions of dependency space, closure system and rough set theory

This section only introduces basic notions, for more extensive introductions refer to [3, 6].

Let $S = (U, AT, V, f)$ be an information system, each subset $B \subseteq AT$ can determine a binary indiscernibility relation

$$Ind(B) = \{(x, y) \in U \times U | \forall m \in B, f(x, m) = f(y, m)\}$$

Let $B, C \subseteq AT$, if $m \in B$ and $Ind(B) \neq Ind(B - m)$, we say m is indispensable; Further if every $m \in B$ is indispensable, we say B is independent. The set of all independent sets of attributes is denoted as IND_S. If $C \subseteq B$, C is independent and $\text{Ind}(B) = Ind(C)$, then C is called a reduct of B. The set of all reducts of B is denoted as $Red_S(B)$. The set of all indispensable attributes in B is called the core of B denoted as $Core_S(B)$. If $Ind(B) \subseteq Ind(C)$, we say $B \rightarrow C$ is a function dependency of S.

Let A be a non-empty and finite set, K is a (union) congruence relation on semilattice $(\mathscr{P}(A), \cup)$ (i.e., K is an indiscernibility relation on $\mathscr{P}(A)$ and closed under union operation), we call (A, K) a dependency space. In that case $\mathscr{P}(A)$ denotes the power-set of A.

Proposition 1 *Let $S = (U, AT, V, f)$ be an information system, we define a binary relation as*:

$$K_S = \{(B, C) \in \mathscr{P}(AT)^2 | Ind(B) = Ind(C)\}$$

Then $D_S = (AT, K_S)$ is a dependency space of information system S.

A closure system on a set P is a set of subsets which contains P and is closed under intersections. Formally: $\mathscr{U} \subseteq \mathscr{P}(P)$ is a closure system if $P \in \mathscr{U}$ and $\mathscr{X} \subseteq \mathscr{U} \Rightarrow \bigcap \mathscr{X} \in \mathscr{U}$.

A closure operator φ on P is a map assigning a closure $\varphi X \subseteq P$ to each subset $X \subseteq P$ under following conditions:

(1) $X \subseteq Y \Rightarrow \varphi X \subseteq \varphi Y$ (2) $X \subseteq \varphi X$

(3) $\varphi\varphi X = \varphi X$

Proposition 2 *If $\mathscr{U}$ is a closure system on P, then*

$$\varphi_{\mathscr{U}}(X) := \bigcap \{A \in \mathscr{U} | X \subseteq A\}$$

is a closure operator on P. Conversely, the set

$$\mathscr{U}_{\varphi} := \{\varphi(X) | X \subseteq P\}$$

of all closures of a closure operator φ is always a closure system.

3 Dependency space and closure system

We can obtain a relatively simple derivative system J_S through pretreatment of an information system S. Actually, J_S is a special information system, and the corresponding formal definition is shown as follows.

Definition 1 Let $S = (U, AT, V, f)$ be an information system with $a \in AT$ and $x, y \in U$, by the following rule

$$(u_i, u_j) I_S a \Leftrightarrow f(u_i, a) = f(u_j, a)$$

S can be transformed into a triple $J_S = (\tilde{U}, AT, I_S)$ with $\tilde{U} = \{(u_i, u_j) | u_i, u_j \in U\}$, we say J_S is the derivative system of S.

Theorem 1 *In $J_S = (\tilde{U}, AT, I_S)$, let $A \subseteq \tilde{U}$* and *$B, B_1, B_2 \subseteq AT$, we define two morphisms*

$$\varphi A = \{m \in AT | (x, y) I_S m, \mathit{for\ all} (x, y) \in A\}$$

$$\psi B = \{(x, y) \in \tilde{U} | (x, y) I_S m, \mathit{for\ all}\ m \in B\}$$

then:

1. $Ind(B) = \psi\ B$
2. *$\varphi\psi$ is a closure operator on AT*
3. *$\mathscr{U}^S_{\varphi\psi} := \{\varphi\psi B | B \subseteq AT\}$ is a closure system on AT*
4. *$D^{\psi}_S = (AT, K^{\psi}_S)$ is a dependency space of S with*
   $$K^{\psi}_S = \{(B_1, B_2) \in \mathscr{P}(AT)^2 | \psi(B_1) = \psi(B_2)\}$$
5. $(B_1, B_2) \in K^{\psi}_S \Leftrightarrow Ind(B_1) = Ind(B_2)$
6. $D^{\psi}_S = D_S$

Proof (1) The statement follows immediately from Definition 1. (2) It is sufficient to show that the conclusion meets all conditions in the definition of a closure operator. Since

$$\begin{aligned}\varphi\psi B &\Longrightarrow \varphi\{(x, y) \in \tilde{U} | (x, y) I_S m, \mathit{for\ all}\ m \in B\} \\ &\Longrightarrow \varphi\{(x, y) \in \tilde{U} | B \subseteq \varphi(x, y)\} \\ &\Longrightarrow \cap \{\varphi(x, y) | B \subseteq \varphi(x, y)\} \supseteq B,\end{aligned}$$

we have $B \subseteq \varphi\psi B$. Taking into account φA and $A_1 \subseteq A_2$, it's obvious that $\varphi A_2 \subseteq \varphi A_1$. Since $A \subseteq \psi\varphi A$ (proved in a similar fashion as $B \subseteq \varphi\psi B$) and $\varphi A_2 \subseteq \varphi A_1$, we can obtain $\varphi\psi\varphi A \subseteq \varphi A$. In addition, because $B \subseteq \varphi\psi B$, we have $\varphi A \subseteq \varphi\psi\varphi A$ (replacing B by φA), and together with $\varphi\psi\varphi A \subseteq \varphi A$, $\varphi\psi\varphi A = \varphi A$ holds, therefore, $\varphi\psi\varphi\psi B = \varphi\psi B$ holds (replacing A by ψ B). Since $\varphi\psi B = \bigcap \{\varphi(x, y) | B \subseteq \varphi(x, y)\}$ and $B \subseteq C$, we have $\varphi\psi B \subseteq \varphi\psi C$.

All together, $\varphi\psi$ is a closure operator on AT. (3) The statement follows immediately from (2) and Proposition 2. (4) The statement follows immediately from (1) and Proposition 1. (5) The statement follows immediately from (1) and (4). (6) The statement follows immediately from (5) and Proposition 1.

Theorem 1 shows that we can generate a closure system $\mathcal{U}^S_{\varphi\psi}$ and a dependency space $D^{\psi}_S = (AT, K^{\psi}_S)$ respectively based on J_S.

4 Introducing dependency space into rough set theory

In this section, we introduce the dependency space in Sect. 3 to solve some essential problems in rough set theory, such as reducts, cores, etc.

It is obvious that K^{ψ}_S is a binary indiscernibility relation on $\mathscr{P}(AT)$. The relation K^{ψ}_S induces a partition denoted as

$\mathscr{P}(AT)/R = \mathscr{P}(AT)/K^{\psi}_S = \{[C]_R | C \in \mathscr{P}(AT)\}$,

where $[C]_R = \{B \in \mathscr{P}(AT) | (C, B) \in K^{\psi}_S\}$.

Theorem 2 *Let $D^{\psi}_S = (AT, K^{\psi}_S)$ be a dependency space, $B \subseteq AT$, then*

1. *$B \in IND_S$ iff $\nexists\, C \in [B]_R$ satisfies $C \subset B$.*
2. *$C \in Red_S(B)$ iff $\nexists\, \tilde{C} \in [B]_R$ satisfies $\tilde{C} \subset C$, where $C \in [B]_R$.*

Theorem 3 *Let $D^{\psi}_S = (AT, K^{\psi}_S)$ be a dependency space of information system S with $B, C \subseteq AT$, then*

$Core_S(B) = \cap\{C | C \in [B]_R \text{ and } C \subseteq B\}$

Proof First, we assume that if $a \in Core_S(B)$, then there exists $C \in [B]_R$ with $C \subseteq B$ satisfying $a \notin C$. From $(B, C) \in K^{\psi}_S$ by $C \in [B]_R$, it follows that $Ind(B) = Ind(C)$ by Theorem 1. Thus from $C \subseteq B - \{a\} \subseteq B$ we can deduce that $Ind(B) = Ind(B - a)$, which contradicts to $a \in Core_S(B)$. Hence if $a \in Core_S(B)$ and the element C satisfying $C \in [B]_R$ and $C \subseteq B$, we have $a \in C$. Conversely, we assume that if $a \in \cap\{C | C \in [B]_R \text{ and } C \subseteq B\}$, then $Ind(B) = Ind(B - \{a\})$ holds. According to Theorem 2, there must exist minimum set $D \in Red_S(B)$. Since $Ind(B) = Ind(B - \{a\})$, obviously, $a \notin D$ holds. And further together with $D \subseteq B$ and $D \in [B]_R$, we can obtain $a \notin \cap\{C | C \in [B]_R \text{ and } C \subseteq B\}$, which is a contradiction, i.e. if $a \in \cap\{C | C \in [B]_R \text{ and } C \subseteq B\}$, then $Ind(B) \neq Ind(B - \{a\})$ holds. Thus, we obtain $a \in Core_S(B)$.

5 Introducing closure system into rough set theory

In this section, inspired by a previous study [6], we will use the closure system in section 3 to solve some essential problems in rough set theory, such as reducts, cores, etc.

Proposition 3 *Let $B, C \subseteq AT$, then following statements hold*

1. $C \subseteq \varphi\psi B \Leftrightarrow \psi B \subseteq \psi C$
2. $B \rightarrow C \Leftrightarrow C \subseteq \varphi\psi B$
3. $Ind(B) = Ind(C) \Leftrightarrow \varphi\psi(B) = \varphi\psi(C)$

Theorem 4 *Let $\mathcal{U}^S_{\varphi\psi}$ be a closure system on AT, $B, C \subseteq AT$, then following statements are equivalent:*

1. *$\varphi\psi C \subseteq \varphi\psi B$ (If $B \subseteq C$, then $\varphi\psi C = \varphi\psi B$)*
2. *For any $L \in \mathcal{U}^S_{\varphi\psi}$, $B \nsubseteq L$ or $C \subseteq L$ holds.*
3. *$B \rightarrow C$*

Proof

(1) $\Longleftrightarrow$ (2). Firstly suppose (1) holds. For any $L \in \mathcal{U}^S_{\varphi\psi}$, if $B \nsubseteq L$, then (2) is true; For any $L \in \mathcal{U}^S_{\varphi\psi}$, if $B \subseteq L$ then $\varphi\psi B \subseteq \varphi\psi L = L$. Since $\varphi\psi C \subseteq \varphi\psi B$, we can obtain $\varphi\psi C \subseteq L \Rightarrow C \subseteq L$, hence (2) is true. Conversely, suppose (2) holds. Obviously, that yields $B \subseteq C \Rightarrow \cap\{L \in \mathcal{U}^S_{\varphi\psi} | C \subseteq L\} \subseteq \cap\{L \in \mathcal{U}^S_{\varphi\psi} | B \subseteq L\} \Rightarrow \varphi\psi C \subseteq \varphi\psi B$ by Proposition 2. Moreover, we can obtain $B \subseteq C \Rightarrow \varphi\psi B \subseteq \varphi\psi C$ from the certification process of Theorem 1, together with $\varphi\psi C \subseteq \varphi\psi B$, we obtain $\varphi\psi C = \varphi\psi B$, hence (1) is true.
(3) $\Longleftrightarrow$ (2). Firstly suppose (3) holds, thus we obtain $C \subseteq \varphi\psi B$ by Proposition 3. For any $L \in \mathcal{U}^S_{\varphi\psi}$, if $B \nsubseteq L$, then (2) is true; For any $L \in \mathcal{U}^S_{\varphi\psi}$, if $B \subseteq L$, then $\varphi\psi B \subseteq \varphi\psi L = L$, and further together with $C \subseteq \varphi\psi B$, we get $C \subseteq L$, therefore, (2) is true. Conversely, suppose (2) holds. In particular, we have that $B \nsubseteq \varphi\psi B$ or $C \subseteq \varphi\psi B$ holds for $\varphi\psi B$. Since $B \subseteq \varphi\psi B$(which denies $B \nsubseteq \varphi\psi B$), we obtain $C \subseteq \varphi\psi B$. And further we can see from Proposition 3 that $B \rightarrow C$, therefore, (3) is true.

Theorem 5 *If $\mathcal{U}^S_{\varphi\psi}$ is a closure system on AT, then*

1. *$IND_S = \{B \subseteq AT | \forall a \in B, B - \{a\} \subseteq L \text{ and } B \nsubseteq L, \exists L \in \mathcal{U}^S_{\varphi\psi}\}$*
2. *$Core_S(B) = \{a \in B | B - \{a\} \subseteq L \text{ and } B \nsubseteq L, \exists L \in \mathcal{U}^S_{\varphi\psi}\}$*

Proof (1) Let $B \subseteq AT$, for any $a \in B$, if there exists $L \in \mathcal{U}^S_{\varphi\psi}$ satisfying $B - \{a\} \subseteq L$ and $B \nsubseteq L$, we can see from Theorem 4 that $\varphi\psi(B) \neq \varphi\psi(B-\{a\})$. And further for any $a \in B, Ind(B) \neq Ind(B-\{a\})$ holds by Proposition 3, thus $B \in IND_S$ is true. (2) Let $B \subseteq AT$ and $a \in B$, if there exists $L \in \mathscr{U}^S_{\varphi\psi}$ satisfying $B-\{a\} \subseteq L$ and $B \nsubseteq L$, we have $\varphi\psi(B) \neq \varphi\psi(B-\{a\})$ by Theorem 4 , and thus $Ind(B) \neq Ind(B-\{a\})$ by Proposition 3, then $a \in Core_S(B)$ is true.

Theorem 6 *Let $\mathscr{U}^S_{\varphi\psi}$ be a closure system on $AT, B, C \subseteq AT$, then $C \in Red_S(B)$iffC is the minimum-subset satisfying the following condition*

$C \cap B \nsubseteq L$orB $\subseteq L$ holds for any $L \in \mathscr{U}^S_{\varphi\psi}$

Proof Let $C \in Red_S(B)$, if C is not the minimum-subset, where C satisfies the condition, then there exists $C_1 \subset C$, and for any $L \in \mathscr{U}^S_{\varphi\psi}$, there exists $C_1 \cap B \nsubseteq L$ or $B \subseteq L$. Since $C_1 \subset C \subseteq B$, we can confirm $C_1 \cap B = C_1$, which means that for any $L \in \mathscr{U}_S$, there is $C_1 \nsubseteq L$ or $B \subseteq L$, and by Theorem 4, we obtain the statement $\varphi\psi C_1 = \varphi\psi B$, and further we can deduce the formula $Ind(B) = Ind(C_1)$ from Proposition 3. Since $C \in Red_S(B)$, we can confirm $Ind(B) = Ind(C)$, which, obviously contradicts with the statement that C is a reduct of B. That is C is the minimum-subset satisfying the condition. Conversely, we provide that C is the minimum-subset, where C satisfies the condition. If $C \nsubseteq B$, then $C_1 = (B \cap C) \subset C$ and $C_1 \cap B = (B \cap C) \cap B = C \cap B \nsubseteq L$, it's clearly that $C_1 \cap B \nsubseteq L$ or $B \subseteq L$ is true for any $L \in \mathscr{U}^S_{\varphi\psi}$, thus contradict with the rule that C is the minimum-subset,where C satisfies the condition, hence we can confirm $C \subseteq B$. We assume $C \notin Red_S(B)$, and then there is $C_1 \subset C \subseteq B$ with $C_1 \in Red_S(B)$. On account of $Ind(B) = Ind(C_1)$, we can obtain $\varphi\psi C_1 = \varphi\psi B$ by Proposition 3 . It's obvious that there exists $C_1 = C_1 \cap B \nsubseteq L$ or $B \subseteq L$ for any $L \in \mathscr{U}^S_{\varphi\psi}$ by Theorem 4. Thus contradict that C is the minimum-set involved in AT, where C satisfies the condition, hence $C \in Red_S(B)$.

6 Relations between closure system and dependency space

This section will reveal some interior relations between closure system $\mathscr{U}^S_{\varphi\psi}$ and dependency space D^{ψ}_S.

Theorem 7 *Let $\mathscr{U}^S_{\varphi\psi}$ be a closure system on AT, then*

$$K(\mathscr{U}^S_{\varphi\psi}) = \{(B,C) | \forall L \in \mathscr{U}^S_{\varphi\psi}, ((B \subseteq L) \text{ and } (C \subseteq L)) \text{ or } ((B \nsubseteq L) \text{ and } (C \nsubseteq L))\}$$

is an indiscernibility relation on $\mathscr{P}(AT)$ and closed under union operation.

Proof Let $(B_1, C_1) \in K(\mathscr{U}^S_{\varphi\psi}), (B_2, C_2) \in K(\mathscr{U}^S_{\varphi\psi})$. Then for any $L \in \mathscr{U}^S_{\varphi\psi}$, we prove as follows: (a) If $B_1 \subseteq L$ and $B_2 \subseteq L$, then $C_1 \subseteq L$ and $C_2 \subseteq L$ hold, and thus $C_1 \cup C_2 \subseteq L$ and $B_1 \cup B_2 \subseteq L$; (b) If $B_1 \subseteq L$ and $B_2 \nsubseteq L$, then we have $C_1 \subseteq L$ and $C_2 \nsubseteq L$, and thus $C_1 \cup C_2 \nsubseteq L$ and $B_1 \cup B_2 \nsubseteq L$; If $B_1 \nsubseteq L$ and $B_2 \subseteq L$ hold, we obtain $C_1 \cup C_2 \nsubseteq L$ and $B_1 \cup B_2 \nsubseteq L$ in the similar fashion; (c) If $B_1 \nsubseteq L$ and $B_2 \nsubseteq L$ hold, then $C_1 \nsubseteq L$ and $C_2 \nsubseteq L$, the further results are $C_1 \cup C_2 \nsubseteq L$ and $B_1 \cup B_2 \nsubseteq L$. All together we have $(B_1 \cup B_2, C_1 \cup C_2) \in K(\mathscr{U}^S_{\varphi\psi})$, hence $K(\mathscr{U}^S_{\varphi\psi})$ is an indiscernibility relation on $\mathscr{P}(AT)$ and closed under union operation.

Theorem 8 *Let $S = (U, AT, V, f)$ be an information system, then following statements hold*

1. *$D(\mathscr{U}^S_{\varphi\psi}) = (AT, K(\mathscr{U}^S_{\varphi\psi}))$ is a dependency space of S*
2. *$D(\mathscr{U}^S_{\varphi\psi}) = D^{\psi}_S = D_S$*

Proof Let $(B,C) \in K^{\psi}_S$, it's obvious that $(B,C) \in K^{\psi}_S \Rightarrow Ind(B) = Ind(C) \Rightarrow \varphi\psi B = \varphi\psi C$. For any $L \in \mathscr{U}^S_{\varphi\psi}$, we discuss in two parts: (a) If $B \subseteq L$, then $C \subseteq \varphi\psi C = \varphi\psi B \subseteq \varphi\psi L = L$, and similarly $B \subseteq L$ yielding from $C \subseteq L$. (b) If $B \nsubseteq L$, then $C \subseteq \varphi\psi C = \varphi\psi B \nsubseteq \varphi\psi L = L$ and similarly $B \nsubseteq L$ yielding from $C \nsubseteq L$. All together we have $(B,C) \in K(\mathscr{U}^S_{\varphi\psi})$. Conversely, suppose $(B,C) \in K(\mathscr{U}^S_{\varphi\psi})$ holds, since $\varphi\psi B \in \mathscr{U}^S_{\varphi\psi}$ and $B \subseteq \varphi\psi B$, we have $C \subseteq \varphi\psi B$, and thus $\varphi\psi C \subseteq \varphi\psi\varphi\psi B \Rightarrow \varphi\psi C \subseteq \varphi\psi B. \varphi\psi B \subseteq \varphi\psi C$ can be proved for the same reason. Hence we conform $\varphi\psi B = \varphi\psi C$. From Proposition 3 and Theorem 1, it follows $(B,C) \in K^{\psi}_S$. Above all, we can obtain $K^{\psi}_S = K(\mathscr{U}^S_{\varphi\psi})$, obviously, which implies $D(\mathscr{U}^S_{\varphi\psi}) = D^{\psi}_S$, and further we can obtain $D(\mathscr{U}^S_{\varphi\psi}) = D^{\psi}_S = D_S$ by Theorem 1.

It's easy to know by Theorem 8 that the dependency space $D(\mathscr{U}^S_{\varphi\psi})$ generated from the closure system $\mathscr{U}^S_{\varphi\psi}$ is equivalent to D^{ψ}_S.

7 Conclusion and further work

This paper is a new achievement in the relation among dependency space, closure system and rough set theory, it mainly provides new solutions to some essential problems of rough set theory. This pater can be divided into the following parts: Firstly, it generates a dependency space and a closure system; second, it solves some essential problems in rough set theory from two different views of dependency space and closure system respectively; finally, it reveals some interior relations between dependency space and closure system. Results will provide a new theoretical basis for analysis and processing of data as well as helping people understands rough set theory in views of dependency space and closure system. How to further introduce these methods into rough set models possessing special relations, such as the variable precision rough set model, probabilistic rough set model, fuzzy rough set model and random sets rough set model, will be our future work.

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