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Fault tolerance in the arrangement graphs ☆

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ABSTRACT

Let n and k be positive integers with $n-k \geqslant 1$. The arrangement graph $A_{n,k}$ is recognized as an attractive interconnection network. Let f_m be the minimum number of faulty vertices that make every sub-arrangement graph $A_{n-m,k-m}$ faulty in $A_{n,k}$ under vertex-failure model. In this paper, we prove that $f_0 = 1$, $f_1 = n$, $f_{n-2} = n!/2$, and $n!/(n-m)! \leqslant f_m \leqslant \binom{k-1}{m-1} n!/(n-m)! - 2\binom{k-2}{m-1} n!/(n-m+1)!$ for $2 \leqslant m \leqslant k-1$.

1. Introduction

The study of interconnection networks has been an important research area for parallel and distributed computer systems. It is well known that an interconnection network is usually represented by an undirected simple graph G. We denote the *vertex set* and the *edge set* of G by $V(G)$ and $E(G)$, respectively. As failures are inevitable, fault-tolerance of interconnection networks has become an important issue and has been extensively studied (see, for example, [4,6,8,10,12–14]).

The fault tolerance of interconnection networks is generally measured by how much of the network structure is preserved in the presence of a given number of vertex and/or edge failures. Parallel algorithms running on the networks utilize the topological properties of these networks. Obviously, in the presence of component failures, the entire interconnection network is not available. Thus the natural question is how large of a *subnetwork* (defined as a smaller network but with the same topological properties as the original one) is still available in the faulty network. Under this consideration, Becker and Simon [4] studied the minimum number of faults, necessary for an adversary to destroy each $(n-k)$-dimensional subcube in an n-dimensional hypercube. Latifi [10] presented a bound on the number of faulty vertices to make every $(n-k)$-dimensional substar faulty in an n-dimensional star graph and also determined the exact value for some special cases. Wang and Yang [12] investigated the minimum number of faulty vertices to make every $(n-k)$-dimensional sub-bubble-sort graph faulty in an n-dimensional bubble-sort graph. Subsequently, this problem was also studied by Wang et al. [13] for k-ary n-cube networks.

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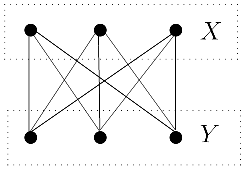

Fig. 1. A balanced bipartite graph.

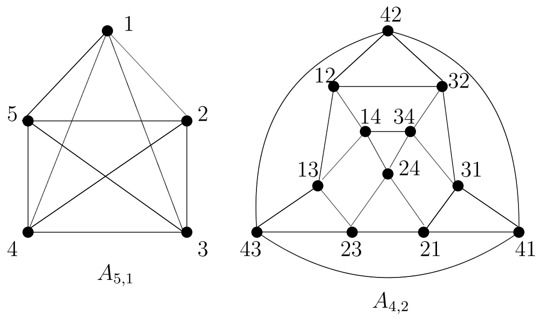

Fig. 2. $A_{5,1}$ and $A_{4,2}$.

The interconnection network considered in this paper is the *arrangement graph*, denoted by $A_{n,k}$, which was proposed by Day and Tripathi [7] as a generalization of the star graph. It is more flexible in its size than the star graph. Since the arrangement graph has been proved to possess many attractive properties such as regularity, vertex symmetry and edge symmetry, it has drawn considerable research attentions recently [5,6,9,11,14]. In this paper, we are interested in the minimum number f_m of faulty vertices to make every sub-arrangement graph $A_{n-m,k-m}$ faulty in $A_{n,k}$ under vertex-failure model. We prove that $f_0 = 1$, $f_1 = n$, $f_{n-2} = n!/2$, and $n!/(n-m)! \leqslant f_m \leqslant \binom{k-1}{m-1} n!/(n-m)! - 2\binom{k-2}{m-1} n!/(n-m+1)!$ for $2 \leqslant m \leqslant k-1$. The rest of this paper is organized as follows. In Section 2, we introduce the arrangement graph and some of its properties. In Section 3, we prove the main results. Conclusions are covered in Section 4.

2. Preliminaries

In the remainder of this paper, we follow [3] for the graph-theoretical terminology and notation not defined here. A graph is called a *balanced bipartite graph* if its vertex set can be partitioned into two subsets X and Y with $|X| = |Y|$ so that every edge has one end in X and one end in Y; such a partition (X, Y) is called a *bipartition* of the graph. Fig. 1 shows the diagram of a balanced bipartite graph. Let G and H be two graphs. G and H are *distinct* if their vertex sets are different. G and H are *disjoint* if their vertex sets have no common vertex. Two edges in $E(G)$ are *independent* if they are nonadjacent in G. Given a positive integer n, let $\langle n\rangle$ denote the set $\{1, 2, \dots, n\}$.

Assume that n and k are two positive integers with $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$. The *vertex set* of the arrangement graph $A_{n,k}$, $V(A_{n,k}) = \{u\colon\ u = u_1u_2\dots u_k$ with $u_i \in \langle n\rangle$ for $1 \leqslant i \leqslant k$ and $u_i \neq u_j$ if $i \neq j\}$, and two vertices $u = u_1u_2\dots u_k$ and $v = v_1v_2\dots v_k$ are *adjacent* if they differ in exactly one position j, where $j \in \langle k\rangle$. Such an edge (u, v) is called a *j-edge*. By definition, $A_{n,k}$ is a regular graph of degree $k(n-k)$ with $n!/(n-k)!$ vertices. Moreover, $A_{n,1}$ is isomorphic to the complete graph K_n. Indeed, $A_{n,n-1}$ is isomorphic to the n-dimensional star graph S_n [1], and $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n [5]. $A_{5,1}$ and $A_{4,2}$ are shown in Fig. 2.

A standard way to view $A_{n,k}$ is via its recursive structure. Let i and j be two positive integers with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$. Let $H_{i,j}$ be the subgraph of $A_{n,k}$ induced by the vertex set $\{u\colon\ u = u_1u_2\dots u_k \in V(A_{n,k})$ and $u_i = j\}$. Then $H_{i,j}$ is isomorphic to $A_{n-1,k-1}$. Thus, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$, and we say that it is a *decomposition* via the i-th position. Every vertex $v = v_1v_2\dots v_k$ in $H_{i,j}$ has exactly $n-k$ neighbors outside of $H_{i,j}$; moreover, its $n-k$ neighbors belong to distinct $H_{i,l}$'s, where $l \in \langle n\rangle \backslash \{v_1, v_2, \dots, v_k\}$. We call these neighbors the *external neighbors* of v. It is easy to see that the edges whose end-vertices belong to distinct $H_{i,j}$'s are i-edges. For a given pair of $H_{i,j}$ and $H_{i,l}$ with $j \neq l$, there are $(n-2)!/(n-k-1)!$ i-edges between them; moreover, these i-edges are independent. For example, $A_{4,2}$ can be decomposed into $H_{2,1}, H_{2,2}, H_{2,3}, H_{2,4}$ via the 2-nd position. For any $j \in \langle 4\rangle$, $H_{2,j}$ is isomorphic to $A_{3,1}$; moreover, $H_{2,1}$, $H_{2,2}$, $H_{2,3}$ and $H_{2,4}$ are the triangles $(21, 31, 41, 21)$, $(12, 32, 42, 12)$, $(13, 23, 43, 13)$ and $(14, 24, 34, 14)$, respectively (see Fig. 2).

Given two integers $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$, for any integer m $(0 \leqslant m \leqslant k-1)$, let $i_1, i_2, \dots, i_m$ be m integers with $1 \leqslant i_1 < i_2 < \dots < i_m \leqslant k$ and let $a_{i_1}, a_{i_2}, \dots, a_{i_m}$ be m pairwise distinct integers in $\langle n\rangle$. Denote $M = \{b_1b_2\dots b_{i_1-1}a_{i_1}b_{i_1+1}\dots b_{i_2-1}a_{i_2}b_{i_2+1}\dots a_{i_m}b_{i_m+1}\dots b_k\colon\ b_1, b_2, \dots, b_{i_1-1}, b_{i_1+1}, \dots, b_{i_2-1}, b_{i_2+1}, \dots, b_{i_m+1}, \dots, b_k \in \langle n\rangle \backslash \{a_{i_1}, a_{i_2}, \dots, a_{i_m}\}$ and they are pairwise distinct$\}$. In particular, $b_1b_2\dots b_{i_1-1}$ and $b_{i_m+1}\dots b_k$ are empty strings if $i_1 = 1$ and $i_m = k$, respectively. Obviously, the subgraph of $A_{n,k}$ induced by M is isomorphic to $A_{n-m,k-m}$. Let X be a *don't care* symbol and let $X^t = \underbrace{XX\ldots X}_{t}$. For the convenience of representation, we denote by a k-length string of symbols $X^{i_1-1}a_{i_1}X^{i_2-i_1-1}a_{i_2}\ldots a_{i_m}X^{k-i_m}$ the subgraph induced by M in $A_{n,k}$. In particular, when $m=0$, $X^{i_1-1}a_{i_1}X^{i_2-i_1-1}a_{i_2}\ldots a_{i_m}X^{k-i_m}$ is just X^k, i.e., $A_{n,k}$.

In addition, we can obtain the following lemma.

Lemma 1. *An $A_{n-m,k-m}$ in $A_{n,k}$ can be uniquely denoted by $X^{i_1-1}a_{i_1}X^{i_2-i_1-1}a_{i_2}\ldots a_{i_m}X^{k-i_m}$.*

Proof. We first prove the following claim.

Claim. *For any $A_{n-k+s,s}$ $(1 \leqslant s \leqslant k)$ in $A_{n,k}$, there exist pairwise distinct $t_1, t_2, \ldots, t_s \in \langle k\rangle$ such that $A_{n-k+s,s}$ contains only $t_1, t_2, \ldots, t_s$-edges.*

By induction on s. We first consider the case that $s=1$. Note that $A_{n-k+1,1}$ is isomorphic to the complete graph K_{n-k+1}. Any two vertices in $A_{n-k+1,1}$ are adjacent. Suppose that $A_{n-k+1,1}$ contains i_1, i_2-edges, where $i_1, i_2 \in \langle k\rangle$ and $i_1 \neq i_2$. Without loss of generality, assume that $i_1 < i_2$. Let $(a_1\ldots a_{i_1}\ldots a_{i_2}\ldots a_k, a_1\ldots a'_{i_1}\ldots a_{i_2}\ldots a_k)$ and $(b_1\ldots b_{i_1}\ldots b_{i_2}\ldots b_k, b_1\ldots b_{i_1}\ldots b'_{i_2}\ldots b_k)$ be an i_1-edge and an i_2-edge in $A_{n-k+1,1}$, respectively. Clearly, either $a_{i_1} \neq b_{i_1}$ or $a'_{i_1} \neq b_{i_1}$, without loss of generality, assume that $a_{i_1} \neq b_{i_1}$. Similarly, either $b_{i_2} \neq a_{i_2}$ or $b'_{i_2} \neq a_{i_2}$, assume that $b_{i_2} \neq a_{i_2}$. By definition, the vertices $a_1\ldots a_{i_1}\ldots a_{i_2}\ldots a_k$ and $b_1\ldots b_{i_1}\ldots b_{i_2}\ldots b_k$ are nonadjacent in $A_{n-k+1,1}$, a contradiction. Thus, the claim holds when $s=1$.

Assume that the claim is true for $s \geqslant 1$. We shall show that the claim holds for $s+1$. For any $A_{n-k+(s+1),s+1}$ in $A_{n,k}$, $E(A_{n-k+(s+1),s+1}) \neq \emptyset$. Without loss of generality, assume that the $A_{n-k+(s+1),s+1}$ contains i-edges. Note that $A_{n-k+(s+1),s+1}$ can be decomposed into $H_{i,j_1}, H_{i,j_2}, \ldots, H_{i,j_{n-k+(s+1)}}$ via the i-th position. For any $j_t \in \{j_1, j_2, \ldots, j_{n-k+(s+1)}\}$, H_{i,j_t} is isomorphic to $A_{n-k+s,s}$. By the induction hypothesis, there exist pairwise distinct $p_1^{j_t}, p_2^{j_t}, \ldots, p_s^{j_t} \in \langle k\rangle\backslash\{i\}$ such that H_{i,j_t} contains only $p_1^{j_t}, p_2^{j_t}, \ldots, p_s^{j_t}$-edges. For any pair of H_{i,j_t} and H_{i,j_q}, we claim that $\{p_1^{j_t}, p_2^{j_t}, \ldots, p_s^{j_t}\} = \{p_1^{j_q}, p_2^{j_q}, \ldots, p_s^{j_q}\}$. Otherwise, there exists $p_l^{j_t}$ $(l \in \langle s\rangle)$ such that H_{i,j_t} contains $p_l^{j_t}$-edges and H_{i,j_q} contains no $p_l^{j_t}$-edge. Then there exists some $t \in \langle n\rangle\backslash\{j_1, j_2, \ldots, j_{n-k+(s+1)}\}$ such that, for any vertex $v_1v_2\ldots v_k \in V(H_{i,j_q})$, $v_{p_l^{j_t}} = t$. For any $u_1u_2\ldots u_k \in V(H_{i,j_t})$, $u_{p_l^{j_t}} \in \{j_1, j_2, \ldots, j_{n-k+(s+1)}\}$ and so $u_{p_l^{j_t}} \neq t$. Note that a vertex in H_{i,j_t} and a vertex in H_{i,j_q} differ in position i for $j_t \neq j_q$. We have that there is no i-edge between H_{i,j_t} and H_{i,j_q}, a contradiction. So $A_{n-k+(s+1),s+1}$ contains only $p_1^{j_t}, p_2^{j_t}, \ldots, p_s^{j_t}, i$-edges. The proof of Claim is complete.

Next, we prove Lemma 1 by induction on m. When $m=0$, $A_{n,k}$ can be uniquely denoted by X^k. Assume that Lemma 1 is true for $m \geqslant 0$. We shall show that Lemma 1 holds for $m+1$. Note that an $A_{n-(m+1),k-(m+1)}$ in $A_{n,k}$ must be in some $A_{n-m,k-m}$. By induction hypothesis, the $A_{n-m,k-m}$ can be uniquely denoted by $X^{i_1-1}a_{i_1}X^{i_2-i_1-1}a_{i_2}\ldots a_{i_m}X^{k-i_m}$. By Claim, there exist pairwise distinct $j_1, j_2, \ldots, j_{k-m} \in \langle k\rangle \setminus \{i_1, i_2, \ldots, i_m\}$ such that $A_{n-(m+1),k-(m+1)}$ contains only $j_1, j_2, \ldots, j_{k-m-1}$-edges and $A_{n-m,k-m}$ contains only $j_1, j_2, \ldots, j_{k-m}$-edges. This implies that the $A_{n-(m+1),k-(m+1)}$ can be obtained by decomposing $X^{i_1-1}a_{i_1}X^{i_2-i_1-1}a_{i_2}\ldots a_{i_m}X^{k-i_m}$ via the j_{k-m}-th position. So there exists $a_{j_{k-m}} \in \langle n\rangle\backslash\{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$ such that, for any vertex $u_1u_2\ldots u_k \in V(A_{n-(m+1),k-(m+1)})$, $u_{j_{k-m}} = a_{j_{k-m}}$. Let $\{t_1, t_2, \ldots, t_{m+1}\} = \{i_1, \ldots, i_m, j_{k-m}\}$ with $t_1 < t_2 < \cdots < t_{m+1}$. Thus the $A_{n-(m+1),k-(m+1)}$ can be uniquely denoted by $X^{t_1-1}a_{t_1}X^{t_2-t_1-1}a_{t_2}\ldots a_{t_{m+1}}X^{k-t_{m+1}}$. The proof of Lemma 1 is complete. □

Lemma 2. *There are $n!/(n-m)!$ disjoint $A_{n-m,k-m}$ and $\binom{k}{m}n!/(n-m)!$ distinct $A_{n-m,k-m}$ in $A_{n,k}$.*

Proof. This lemma is trivial when $m=0$. In the following, assume that $m \geqslant 1$. By Lemma 1, an $A_{n-m,k-m}$ in $A_{n,k}$ can be uniquely denoted by $X^{i_1-1}a_{i_1}X^{i_2-i_1-1}a_{i_2}\ldots a_{i_m}X^{k-i_m}$, where $i_1, i_2, \ldots, i_m$ are m integers with $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant k$ and $a_{i_1}, a_{i_2}, \ldots, a_{i_m} \in \langle n\rangle$ are pairwise distinct. According to the values of $i_1, i_2, \ldots, i_m$, we divide all the distinct $A_{n-m,k-m}$ in $A_{n,k}$ into $\binom{k}{m}$ sets $\Omega_1, \Omega_2, \ldots, \Omega_{\binom{k}{m}}$, where, for every $p \in \{1, 2, \ldots, \binom{k}{m}\}$, $\Omega_p = \{X^{i_1^p-1}a_{i_1^p}X^{i_2^p-i_1^p-1}a_{i_2^p}\ldots a_{i_m^p}X^{k-i_m^p}$: $a_{i_1^p}, a_{i_2^p}, \ldots, a_{i_m^p} \in \langle n\rangle$ are pairwise distinct$\}$. Note that any two distinct $A_{n-m,k-m}$ $X^{i_1^p-1}a_{i_1^p}X^{i_2^p-i_1^p-1}a_{i_2^p}\ldots a_{i_m^p}X^{k-i_m^p}$ and $X^{i_1^p-1}b_{i_1^p}X^{i_2^p-i_1^p-1}b_{i_2^p}\ldots b_{i_m^p}X^{k-i_m^p}$ in Ω_p have no common vertex because there is some $l \in \{i_1^p, i_2^p, \ldots, i_m^p\}$ such that $a_l \neq b_l$. So $X^{i_1^p-1}a_{i_1^p}X^{i_2^p-i_1^p-1}a_{i_2^p}\ldots a_{i_m^p}X^{k-i_m^p}$ and $X^{i_1^p-1}b_{i_1^p}X^{i_2^p-i_1^p-1}b_{i_2^p}\ldots b_{i_m^p}X^{k-i_m^p}$ are disjoint. It follows that there are $|\Omega_p| = n!/(n-m)!$ disjoint $A_{n-m,k-m}$ in $A_{n,k}$.

For two distinct integers $p, q \in \{1, 2, \ldots, \binom{k}{m}\}$, any two $A_{n-m,k-m}$ $X^{i_1^p-1}a_{i_1^p}X^{i_2^p-i_1^p-1}a_{i_2^p}\ldots a_{i_m^p}X^{k-i_m^p}$ in Ω_p and $X^{i_1^q-1}a_{i_1^q}X^{i_2^q-i_1^q-1}a_{i_2^q}\ldots a_{i_m^q}X^{k-i_m^q}$ in Ω_q are distinct, since otherwise the vertex set of $X^{i_1^p-1}a_{i_1^p}X^{i_2^p-i_1^p-1}a_{i_2^p}\ldots a_{i_m^p}X^{k-i_m^p}$ and the vertex

Table 1
The faulty vertices and damaged $A_{n-1,k-1}$ in $A_{n,k}$.

Faulty vertices	Damaged $A_{n-1,k-1}$
$12\ldots(k-1)k$	$1X^{k-1},\ X2X^{k-2},\ \ldots,\ X^{k-2}(k-1)X,\ X^{k-1}k$
$23\ldots k(k+1)$	$2X^{k-1},\ X3X^{k-2},\ \ldots,\ X^{k-2}kX,\ X^{k-1}(k+1)$
$\vdots$	$\vdots \quad \vdots \quad \ldots \quad \vdots \quad \vdots$
$n1\ldots(k-2)(k-1)$	$nX^{k-1},\ X1X^{k-2},\ \ldots,\ X^{k-2}(k-2)X,\ X^{k-1}(k-1)$

set of $X^{i_1^q-1}a_{i_1^q}X^{i_2^q-i_1^q-1}a_{i_2^q}\ldots a_{i_m^q}X^{k-i_m^q}$ are the same, which yields that $i_1^p=i_1^q$, $i_2^p=i_2^q$, $\ldots$, $i_m^p=i_m^q$, a contradiction. Therefore, there are $\sum_{p=1}^{\binom{k}{m}}|\Omega_p|=\binom{k}{m}n!/(n-m)!$ distinct $A_{n-m,k-m}$ in $A_{n,k}$. The proof of Lemma 2 is complete. □

3. Enumeration of faulty vertices

3.1. The lower and upper bounds

Given two integers n and k with $n\geqslant 2$ and $1\leqslant k\leqslant n-1$, we are interested in finding f_m, the minimum number of faulty vertices to make every sub-arrangement graph $A_{n-m,k-m}$ faulty in $A_{n,k}$ under vertex-failure model, where $0\leqslant m\leqslant k-1$.

Lemma 3. $n!/(n-m)!\leqslant f_m\leqslant\binom{k}{m}n!/(n-m)!$.

Proof. By Lemma 2, $A_{n,k}$ can be divided into $n!/(n-m)!$ disjoint $A_{n-m,k-m}$. To damage all the disjoint $A_{n-m,k-m}$ in $A_{n,k}$, we need at least one faulty vertex for each $A_{n-m,k-m}$, which implies that $f_m\geqslant n!/(n-m)!$.

The upper bound on f_m can be obtained by making a vertex faulty in each of the $\binom{k}{m}n!/(n-m)!$ distinct $A_{n-m,k-m}$ in $A_{n,k}$. This will render: $f_m\leqslant\binom{k}{m}n!/(n-m)!$. Combining this with the fact that $f_m\geqslant n!/(n-m)!$, the lemma holds. □

Lemma 4. *(See [1].) The n-dimensional star graph is a balanced bipartite graph.*

The following theorem gives the exact value of f_m for some special cases.

Theorem 1. *Denote by f_m the minimum number of faulty vertices to make every sub-arrangement graph $A_{n-m,k-m}$ faulty in $A_{n,k}$ under vertex-failure model. Then the following results hold.*

(1) $f_0=1$.
(2) $f_1=n$.
(3) $f_{n-2}=n!/2$.

Proof. (1) Since the failure of a single vertex will damage the $A_{n,k}$, we have $f_0\leqslant 1$. Lemma 3 implies that $f_0\geqslant n!/(n-0)!=1$. So $f_0=1$.

(2) By Lemma 2, there are kn distinct $A_{n-1,k-1}$ in $A_{n,k}$. For $0\leqslant i\leqslant n-1$ and $1\leqslant p\leqslant k$, define v_p^i as follows:

$$v_p^i=\begin{cases}p+i, & \text{if } p+i\leqslant n;\\ p+i-n, & \text{if } p+i>n.\end{cases}$$

Note that the vertex $v_1^iv_2^i\ldots v_k^i$ will damage the k distinct $A_{n-1,k-1}$ $v_1^iX^{k-1}$, $Xv_2^iX^{k-2}$, $\ldots$, $X^{k-2}v_{k-1}^iX$ and $X^{k-1}v_k^i$. Combining this with the fact that $\{v_l^0,v_l^1,\ldots,v_l^{n-1}\}=\langle n\rangle$ for any $l\in\langle k\rangle$, the vertices $v_1^0v_2^0\ldots v_k^0$, $v_1^1v_2^1\ldots v_k^1$, $\ldots$, $v_1^{n-1}v_2^{n-1}\ldots v_k^{n-1}$ (i.e., $12\ldots k$, $23\ldots(k+1)$, $\ldots$, $n1\ldots(k-1)$) will damage every $A_{n-1,k-1}$ in $A_{n,k}$, which implies that $f_1\leqslant n$. See Table 1 for more details. Lemma 3 implies that $f_1\geqslant n!/(n-1)!=n$. So $f_1=n$.

(3) Note that f_{n-2} is the minimum number of faulty vertices to make every $A_{n-(n-2),k-(n-2)}$ faulty in $A_{n,k}$. We have that $k-(n-2)\geqslant 1$, i.e., $k\geqslant n-1$. Combining this with the fact that $1\leqslant k\leqslant n-1$, $k=n-1$. By definition, $A_{n-(n-2),(n-1)-(n-2)}$ (i.e., $A_{2,1}$) is just an edge. Recall that $A_{n,n-1}$ is isomorphic to the n-dimensional star graph. By Lemma 4, $A_{n,n-1}$ is a balanced bipartite graph. Let (X,Y) be a bipartition of $A_{n,n-1}$. Then $|X|=|Y|=n!/2$ (see, for example, Fig. 3). Now, every failing vertex in X will ensure every $A_{2,1}$ in $A_{n,n-1}$ faulty, which implies that $f_{n-2}\leqslant n!/2$. By Lemma 3, $f_{n-2}\geqslant n!/(n-(n-2))!=n!/2$. So $f_{n-2}=n!/2$. □

Note that for some special cases the exact value of f_m coincides with the lower bound. But there is a large gap between the lower bound $n!/(n-m)!$ and the upper bound $\binom{k}{m}n!/(n-m)!$ in Lemma 3. In the following, we shall improve the upper bound on f_m.

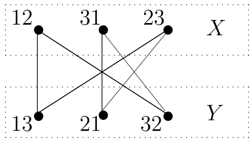

Fig. 3. A bipartition of $A_{3,2}$.

Table 2
The faulty vertices and damaged $A_{n-2,k-2}$ in Θ_1.

Faulty vertices	Damaged $A_{n-2,k-2}$			
$123\dots(k-1)k$	$12X^{k-2}$,	$1X3X^{k-3}$,	$\dots$,	$1X^{k-2}k$
$134\dots k(k+1)$	$13X^{k-2}$,	$1X4X^{k-3}$,	$\dots$,	$1X^{k-2}(k+1)$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
$1n2\dots(k-2)(k-1)$	$1nX^{k-2}$,	$1X2X^{k-3}$,	$\dots$,	$1X^{k-2}(k-1)$
$213\dots(k-1)k$	$21X^{k-2}$,	$2X3X^{k-3}$,	$\dots$,	$2X^{k-2}k$
$234\dots k(k+1)$	$23X^{k-2}$,	$2X4X^{k-3}$,	$\dots$,	$2X^{k-2}(k+1)$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
$2n1\dots(k-2)(k-1)$	$2nX^{k-2}$,	$2X1X^{k-3}$,	$\dots$,	$2X^{k-2}(k-1)$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
$n12\dots(k-2)(k-1)$	$n1X^{k-2}$,	$nX2X^{k-3}$,	$\dots$,	$nX^{k-2}(k-1)$
$n23\dots(k-1)k$	$n2X^{k-2}$,	$nX3X^{k-3}$,	$\dots$,	$nX^{k-2}k$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
$n(n-1)1\dots(k-3)(k-2)$	$n(n-1)X^{k-2}$,	$nX1X^{k-3}$,	$\dots$,	$nX^{k-2}(k-2)$

3.2. A better upper bound on f_m

We first present a better upper bound on f_2 by giving a set of faulty vertices that damage all $A_{n-2,k-2}$ in $A_{n,k}$, which yields that $k-2 \geqslant 1$, i.e., $k \geqslant 3$.

For any $u = u_1u_2\dots u_k \in V(A_{n,k})$ and $i \in \langle k-1\rangle$, define a mapping Φ_i as follows: $\Phi_i(u) = \{u_1u_2\dots u_{i-1}u_iu_{i+1}\dots u_k, u_1^1u_2^1\dots u_{i-1}^1u_iu_{i+1}^1\dots u_k^1, u_1^2u_2^2\dots u_{i-1}^2u_iu_{i+1}^2\dots u_k^2, \dots, u_1^{n-2}u_2^{n-2}\dots u_{i-1}^{n-2}u_iu_{i+1}^{n-2}\dots u_k^{n-2}\colon u_lu_l^1u_l^2\dots u_l^{n-2}$ is a cyclic permutation of $12\dots n$ missing u_i for every $l \in \langle k\rangle\backslash\{i\}\}$. We denote the vertices $u_1u_2\dots u_{i-1}u_iu_{i+1}\dots u_k$ and $u_1^ju_2^j\dots u_{i-1}^ju_iu_{i+1}^j\dots u_k^j$ ($j \in \langle n-2\rangle$) by $\phi_{i,0}(u)$ and $\phi_{i,j}(u)$, respectively. For example, let $u = 1234 \in V(A_{5,4})$, then $\Phi_2(u) = \{1234, 3245, 4251, 5213\}$ (1345, 3451 and 4513 are cyclic permutations of 12345 missing $u_2 = 2$), and $\phi_{2,0}(u) = 1234$, $\phi_{2,1}(u) = 3245$, $\phi_{2,2}(u) = 4251$, $\phi_{2,3}(u) = 5213$.

Lemma 5. *$f_2 \leqslant \binom{k-1}{1}n(n-1) - 2\binom{k-2}{1}n$ for $k \geqslant 3$.*

Proof. By Lemma 2, there are $\binom{k}{2}n(n-1)$ distinct $A_{n-2,k-2}$ in $A_{n,k}$. Divide all the distinct $A_{n-2,k-2}$ in $A_{n,k}$ into $k-1$ sets $\Theta_1, \Theta_2, \dots, \Theta_{k-1}$, where, for $i \in \langle k-1\rangle$, $\Theta_i = \{X^{i-1}a_iX^{j-i-1}a_jX^{k-j}\colon i+1 \leqslant j \leqslant k$ and $a_i, a_j \in \langle n\rangle$ are distinct$\}$. Clearly, $|\Theta_i| = \binom{k-i}{1}n(n-1) = (k-i)n(n-1)$ for $i \in \langle k-1\rangle$. For any two integers $1 \leqslant p \leqslant k-1$ and $1 \leqslant q \leqslant n$, define the vertex $u_{p,q} = u_1u_2\dots u_{p-1}qu_{p+1}\dots u_k$ as follows:

(*a*) when $1 \leqslant q \leqslant k-p$, $u_1u_2\dots u_{p-1}$ is the permutation $(n-p+2)(n-p+3)\dots n$ and $u_{p+1}\dots u_k$ is the permutation $12\dots(k-p+1)$ missing q. In particular, $u_1u_2\dots u_{p-1}$ is an empty string if $p = 1$;

(*b*) when $k-p+1 \leqslant q \leqslant n-p+1$, $u_1u_2\dots u_{p-1}$ is the permutation $(n-p+2)(n-p+3)\dots n$ and $u_{p+1}\dots u_k$ is the permutation $12\dots(k-p)$;

(*c*) when $n-p+2 \leqslant q \leqslant n$, $u_1u_2\dots u_{p-1}$ is the permutation $(n-p+1)(n-p+2)\dots n$ missing q and $u_{p+1}\dots u_k$ is the permutation $12\dots(k-p)$.

Then the vertex set $\Phi_1(u_{1,1}) \cup \Phi_1(u_{1,2}) \cup \dots \cup \Phi_1(u_{1,n})$ can damage all the $A_{n-2,k-2}$ in Θ_1. See Table 2 for more details. Note that $|\Phi_1(u_{1,1}) \cup \Phi_1(u_{1,2}) \cup \dots \cup \Phi_1(u_{1,n})| = n(n-1)$. Now we find $n(n-1)$ faulty vertices to damage all the $A_{n-2,k-2}$ in Θ_1.

The vertex set $\Phi_2(u_{2,1}) \cup \Phi_2(u_{2,2}) \cup \dots \cup \Phi_2(u_{2,n})$ can damage all the $A_{n-2,k-2}$ in Θ_2. See Table 3 for more details. Note that $|\Phi_2(u_{2,1}) \cup \Phi_2(u_{2,2}) \cup \dots \cup \Phi_2(u_{2,n})| = n(n-1)$. Now we find $n(n-1)$ faulty vertices to damage all the $A_{n-2,k-2}$ in Θ_2.

For $3 \leqslant i \leqslant k-1$, the vertex set $\Phi_i(u_{i,1}) \cup \Phi_i(u_{i,2}) \cup \dots \cup \Phi_i(u_{i,n})$ can damage all the $A_{n-2,k-2}$ in Θ_i, and so we find $n(n-1)$ faulty vertices to damage all the $A_{n-2,k-2}$ in Θ_i. See Table 4 for more details about Θ_{k-1}.

For any integer $2 \leqslant j \leqslant k-1$, it is not hard to verify that the following results hold.

Table 3
The faulty vertices and damaged $A_{n-2,k-2}$ in Θ_2.

Faulty vertices	Damaged $A_{n-2,k-2}$			
$n123\ldots(k-1)$	$X12X^{k-3}$,	$X1X3X^{k-4}$,	$\ldots$,	$X1X^{k-3}(k-1)$
$2134\ldots k$	$X13X^{k-3}$,	$X1X4X^{k-4}$,	$\ldots$,	$X1X^{k-3}k$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
$(n-1)1n2\ldots(k-2)$	$X1nX^{k-3}$,	$X1X2X^{k-4}$,	$\ldots$,	$X1X^{k-3}(k-2)$
$n213\ldots(k-1)$	$X21X^{k-3}$,	$X2X3X^{k-4}$,	$\ldots$,	$X2X^{k-3}(k-1)$
$1234\ldots k$	$X23X^{k-3}$,	$X2X4X^{k-4}$,	$\ldots$,	$X2X^{k-3}k$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
$(n-1)2n1\ldots(k-2)$	$X2nX^{k-3}$,	$X2X1X^{k-4}$,	$\ldots$,	$X2X^{k-3}(k-2)$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
$(n-1)n12\ldots(k-2)$	$Xn1X^{k-3}$,	$XnX2X^{k-4}$,	$\ldots$,	$XnX^{k-3}(k-2)$
$1n23\ldots(k-1)$	$Xn2X^{k-3}$,	$XnX3X^{k-4}$,	$\ldots$,	$XnX^{k-3}(k-1)$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
$(n-2)n(n-1)1\ldots(k-3)$	$Xn(n-1)X^{k-3}$,	$XnX1X^{k-4}$,	$\ldots$,	$XnX^{k-3}(k-3)$

Table 4
The faulty vertices and damaged $A_{n-2,k-2}$ in Θ_{k-1}.

Faulty vertices	Damaged $A_{n-2,k-2}$
$(n-k+3)(n-k+4)\ldots n12$	$X^{k-2}12$
$(n-k+4)(n-k+5)\ldots 213$	$X^{k-2}13$
$\vdots$	$\vdots$
$(n-k+2)(n-k+3)\ldots(n-1)1n$	$X^{k-2}1n$
$(n-k+3)(n-k+4)\ldots n21$	$X^{k-2}21$
$(n-k+4)(n-k+5)\ldots 123$	$X^{k-2}23$
$\vdots$	$\vdots$
$(n-k+2)(n-k+3)\ldots(n-1)2n$	$X^{k-2}2n$
$\ldots$	$\ldots$
$(n-k+2)(n-k+3)\ldots(n-1)n1$	$X^{k-2}n1$
$(n-k+3)(n-k+4)\ldots 1n2$	$X^{k-2}n2$
$\vdots$	$\vdots$
$(n-k+1)(n-k+2)\ldots(n-2)n(n-1)$	$X^{k-2}n(n-1)$

(a) $\phi_{j,0}(u_{j,1}) \in \Phi_{j-1}(u_{j-1,n})$ and $\phi_{j,1}(u_{j,1}) \in \Phi_{j-1}(u_{j-1,2})$.
(b) For any integer $2 \leqslant i \leqslant n-2$, $\phi_{j,i-1}(u_{j,i}) \in \Phi_{j-1}(u_{j-1,i-1})$ and $\phi_{j,i}(u_{j,i}) \in \Phi_{j-1}(u_{j-1,i+1})$.
(c) $\phi_{j,n-2}(u_{j,n-1}) \in \Phi_{j-1}(u_{j-1,n-2})$ and $\phi_{j,0}(u_{j,n-1}) \in \Phi_{j-1}(u_{j-1,n})$.
(d) $\phi_{j,0}(u_{j,n}) \in \Phi_{j-1}(u_{j-1,n-1})$ and $\phi_{j,1}(u_{j,n}) \in \Phi_{j-1}(u_{j-1,1})$.

So, given an integer $2 \leqslant j \leqslant k-1$, there exist at least two vertices in both $\Phi_j(u_{j,i})$ and $\Phi_{j-1}(u_{j-1,1}) \cup \Phi_{j-1}(u_{j-1,2}) \cup \cdots \cup \Phi_{j-1}(u_{j-1,n})$ for every $i \in \langle n \rangle$. Thus we have found $|\bigcup_{j=1}^{k-1}(\Phi_j(u_{j,1}) \cup \Phi_j(u_{j,2}) \cup \cdots \cup \Phi_j(u_{j,n}))| \leqslant (k-1)n(n-1) - 2(k-2)n$ faulty vertices to damage all the $A_{n-2,k-2}$ in $\Theta_1 \cup \Theta_2 \cup \cdots \cup \Theta_{k-1}$, which implies that $f_2 \leqslant (k-1)n(n-1) - 2(k-2)n = \binom{k-1}{1}n(n-1) - 2\binom{k-2}{1}n$. The proof is complete. □

More generally, we give a better upper bound on f_m. The following lemma is useful.

Lemma 6. *(See [2].) Let s, t be two nonnegative integers with $s \geqslant t$. Then $\binom{s-1}{t-1} + \binom{s-2}{t-1} + \cdots + \binom{t-1}{t-1} = \binom{s}{t}$.*

Theorem 2. *Denote by f_m the minimum number of faulty vertices to make every sub-arrangement graph $A_{n-m,k-m}$ faulty in $A_{n,k}$ under vertex-failure model, where $2 \leqslant m \leqslant k-1$. Then $f_m \leqslant \binom{k-1}{m-1}n!/(n-m)! - 2\binom{k-2}{m-1}n!/(n-m+1)!$ for $k \geqslant 3$.*

Proof. We prove the theorem by induction on m. By Lemma 5, the theorem holds for $m = 2$. Assume the theorem holds for $m-1$ ($m \geqslant 3$), i.e., $f_{m-1} \leqslant \binom{k-1}{m-2}n!/(n-m+1)! - 2\binom{k-2}{m-2}n!/(n-m+2)!$ for $k \geqslant 3$. We shall show that the theorem holds for m. By Lemma 2, there are $\binom{k}{m}n!/(n-m)!$ distinct $A_{n-m,k-m}$ in $A_{n,k}$. Divide all the distinct $A_{n-m,k-m}$ in $A_{n,k}$ into $k-m+1$ sets $\Theta_1, \Theta_2, \ldots, \Theta_{k-m+1}$, where, for every $i \in \langle k-m+1\rangle$, $\Theta_i = \{X^{i-1}a_iX^{i_2-i-1}a_{i_2}X^{i_3-i_2-1}a_{i_3}\ldots a_{i_m}X^{k-i_m}\colon\ i+1 \leqslant i_2 < i_3 < \cdots < i_m \leqslant k$ and $a_i, a_{i_2}, a_{i_3}, \ldots, a_{i_m} \in \langle n\rangle$ are pairwise distinct$\}$ and $|\Theta_i| = \binom{k-i}{m-1}n!/(n-m)!$. By Lemma 6, $|\Theta_1| + |\Theta_2| + \cdots + |\Theta_{k-m+1}| = (\binom{k-1}{m-1} + \binom{k-2}{m-1} + \cdots + \binom{m-1}{m-1})n!/(n-m)! = \binom{k}{m}n!/(n-m)!$. For every $i \in \langle k-m+1\rangle$, denote by W_i the set of faulty vertices with the minimum cardinality to damage all $A_{n-m,k-m}$ in Θ_i. For some $r \in \langle n\rangle$ and every $i \in \langle k-m+1\rangle$, we denote by $\Theta_i(r)$ the set $\{X^{i-1}rX^{i_2-i-1}a_{i_2}X^{i_3-i_2-1}a_{i_3}\ldots a_{i_m}X^{k-i_m}\colon\ 1 \leqslant i < i_2 < i_3 < \cdots < i_m \leqslant k$ and $a_{i_2}, a_{i_3}, \ldots, a_{i_m} \in \langle n\rangle\backslash\{r\}$ are pairwise distinct$\}$, and denote by $W_i(r)$ the set of faulty vertices with the minimum cardinality to damage all $A_{n-m,k-m}$ in $\Theta_i(r)$. Clearly, for any $i \in \langle k-m+1\rangle$, $\bigcup_{r=1}^{n} W_i(r) = W_i$ and $W_i(r) \cap W_i(r') = \emptyset$, where $r, r' \in \langle n\rangle$ and $r \neq r'$.

First, we shall determine the value of $|W_1|$. Denote $\Theta'_1 = \{X^{i_2-2}a_{i_2}X^{i_3-i_2-1}a_{i_3}\ldots a_{i_m}X^{k-i_m}\colon\ 2 \leqslant i_2 < i_3 < \cdots < i_m \leqslant k$ and $a_{i_2}, a_{i_3}, \ldots, a_{i_m} \in \langle n\rangle\backslash\{r\}$ are pairwise distinct$\}$. Note that Θ'_1 is the set of all the $A_{(n-1)-(m-1),(k-1)-(m-1)}$ in $A_{n-1,k-1}$. By the induction hypothesis, there exists a set W'_1 of faulty vertices to damage all the $A_{(n-1)-(m-1),(k-1)-(m-1)}$ in $A_{n-1,k-1}$ with $|W'_1| \leqslant \binom{k-2}{m-2}(n-1)!/(n-m)! - 2\binom{k-3}{m-2}(n-1)!/(n-m+1)!$. Thus the faulty vertices in W'_1 damage all the $A_{(n-1)-(m-1),(k-1)-(m-1)}$ in Θ'_1. Let $W^*_1(r) = \{ra_2a_3\ldots a_k \in V(A_{n,k})\colon\ a_2a_3\ldots a_k \in W'_1\}$ be a set of faulty vertices. Then $|W^*_1(r)| = |W'_1|$. Recall that $\Theta_1(r) = \{rX^{i_2-2}a_{i_2}X^{i_3-i_2-1}a_{i_3}\ldots a_{i_m}X^{k-i_m}\colon\ 2 \leqslant i_2 < i_3 < \cdots < i_m \leqslant k$ and $a_{i_2}, a_{i_3}, \ldots, a_{i_m} \in \langle n\rangle\backslash\{r\}$ are pairwise distinct$\}$. The faulty vertices in $W^*_1(r)$ damage all the $A_{n-m,k-m}$ in $\Theta_1(r)$. Therefore $|W_1(r)| \leqslant |W^*_1(r)| = |W'_1| \leqslant \binom{k-2}{m-2}(n-1)!/(n-m)! - 2\binom{k-3}{m-2}(n-1)!/(n-m+1)!$. Note that for any $r' \in \langle n\rangle \setminus \{r\}$, $|W_1(r')| = |W_1(r)|$. It follows that $|W_1| = \sum_{r=1}^{n} |W_1(r)| \leqslant n(\binom{k-2}{m-2}(n-1)!/(n-m)! - 2\binom{k-3}{m-2}(n-1)!/(n-m+1)!) = \binom{k-2}{m-2}n!/(n-m)! - 2\binom{k-3}{m-2}n!/(n-m+1)!$.

Next, we shall determine the value of $|W_2|$. Denote $\Theta'_2 = \{X^{i_2-3}a_{i_2}X^{i_3-i_2-1}a_{i_3}\ldots a_{i_m}X^{k-i_m}\colon\ 3 \leqslant i_2 < i_3 < \cdots < i_m \leqslant k$ and $a_{i_2}, a_{i_3}, \ldots, a_{i_m} \in \langle n\rangle\backslash\{r\}$ are pairwise distinct$\}$. Note that Θ'_2 is the set of all the $A_{(n-1)-(m-1),(k-2)-(m-1)}$ in $A_{n-1,k-2}$. By the induction hypothesis, there exists a set W'_2 of faulty vertices to damage all the $A_{(n-1)-(m-1),(k-2)-(m-1)}$ in $A_{n-1,k-2}$ with $|W'_2| \leqslant \binom{k-3}{m-2}(n-1)!/(n-m)! - 2\binom{k-4}{m-2}(n-1)!/(n-m+1)!$. Thus the faulty vertices in W'_2 damage all the $A_{(n-1)-(m-1),(k-2)-(m-1)}$ in Θ'_2. Let $W^*_2(r) = \{ara_3a_4\ldots a_k \in V(A_{n,k})\colon\ a_3a_4\ldots a_k \in W'_2$ and a is some integer in $\langle n\rangle\backslash\{r, a_3, a_4, \ldots, a_k\}\}$ be a set of faulty vertices. Then $|W^*_2(r)| = |W'_2|$. Recall that $\Theta_2(r) = \{XrX^{i_2-3}a_{i_2}X^{i_3-i_2-1}a_{i_3}\ldots a_{i_m} X^{k-i_m}\colon\ 3 \leqslant i_2 < i_3 < \cdots < i_m \leqslant k$ and $a_{i_2}, a_{i_3}, \ldots, a_{i_m} \in \langle n\rangle\backslash\{r\}$ are pairwise distinct$\}$. The faulty vertices in $W^*_2(r)$ damage all the $A_{n-m,k-m}$ in $\Theta_2(r)$. Therefore $|W_2(r)| \leqslant |W^*_2(r)| = |W'_2| \leqslant \binom{k-3}{m-2}(n-1)!/(n-m)! - 2\binom{k-4}{m-2}(n-1)!/(n-m+1)!$. Note that for any $r' \in \langle n\rangle \setminus \{r\}$, $|W_2(r')| = |W_2(r)|$. It follows that $|W_2| = \sum_{r=1}^{n} |W_2(r)| \leqslant n(\binom{k-3}{m-2}(n-1)!/(n-m)! - 2\binom{k-4}{m-2}(n-1)!/(n-m+1)!) = \binom{k-3}{m-2}n!/(n-m)! - 2\binom{k-4}{m-2}n!/(n-m+1)!$.

We proceed in a similar way until we get $|W_{k-m}| \leqslant \binom{m-1}{m-2}n!/(n-m)! - 2\binom{m-2}{m-2}n!/(n-m+1)!$.

Now, we consider the remaining $\Theta_{k-m+1} = \{X^{k-m}a_{k-m+1}a_{k-m+2}\ldots a_k\colon a_{k-m+1}, a_{k-m+2}, \ldots, a_k \in \langle n\rangle$ are pairwise distinct$\}$. There are $n!/(n-m)!$ disjoint $A_{n-m,k-m}$ in Θ_{k-m+1}. By making a vertex faulty in each of the $n!/(n-m)!$ disjoint $A_{n-m,k-m}$ in Θ_{k-m+1}, we have $|W_{k-m+1}| = n!/(n-m)! = \binom{m-2}{m-2}n!/(n-m)!$. Therefore,

$$\begin{aligned}
f_m &= |W_1| + |W_2| + \cdots + |W_{k-m}| + |W_{k-m+1}| \\
&\leqslant \binom{k-2}{m-2}n!/(n-m)! - 2\binom{k-3}{m-2}n!/(n-m+1)! \\
&\quad + \binom{k-3}{m-2}n!/(n-m)! - 2\binom{k-4}{m-2}n!/(n-m+1)! + \cdots \\
&\quad + \binom{m-1}{m-2}n!/(n-m)! - 2\binom{m-2}{m-2}n!/(n-m+1)! + \binom{m-2}{m-2}n!/(n-m)! \\
&= \left(\binom{k-2}{m-2} + \binom{k-3}{m-2} + \cdots + \binom{m-2}{m-2}\right)n!/(n-m)! \\
&\quad - 2\left(\binom{k-3}{m-2} + \binom{k-4}{m-2} + \cdots + \binom{m-2}{m-2}\right)n!/(n-m+1)!.
\end{aligned}$$

By Lemma 6, $f_m \leqslant \binom{k-1}{m-1}n!/(n-m)! - 2\binom{k-2}{m-1}n!/(n-m+1)!$. The proof is complete. □

4. Conclusions

In this paper, we investigate f_m, the minimum number of failing vertices which make every sub-arrangement graph $A_{n-m,k-m}$ faulty in an arrangement graph $A_{n,k}$ under vertex-failure model. We present the lower and upper bounds on f_m, and determine the exact value of f_m for some special cases. The results can be used in the reliability analysis of the subnetworks in the arrangement graphs. Determination of the exact value of f_m remains an open problem for the general case.

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