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Fault tolerance in k-ary n-cube networks[☆]

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ABSTRACT

The k-ary n-cube Q_n^k is one of the most commonly used interconnection topologies for parallel and distributed computing systems. Let $f(n, m)$ be the minimum number of faulty nodes that make every $(n-m)$-dimensional subcube Q_{n-m}^k faulty in Q_n^k under node-failure models. In this paper, we prove that $f(n, 0) = 1, f(n, 1) = k$ for odd $k \geq 3, f(n, n-1) = k^{n-1}$ for odd $k \geq 3$, and $k^m \leq f(n, m) \leq \binom{n-1}{m-1} k^m - \binom{n-2}{m-1} k^{m-1}$ for odd $k \geq 3$.

1. Introduction

In many parallel computer systems, processors are connected based on an interconnection network. Popular instances of interconnection networks include hypercubes [2,5,7], star graphs [8,10,16], bubble-sort graphs [17], and k-ary n-cubes [1,11,15,18]. It is well known that an interconnection network is usually represented by an undirected simple graph G. We denote the node set and the link set of G by $V(G)$ and $E(G)$, respectively.

In a large-scale multiprocessor system, failures of components are inevitable. Thus, fault tolerance of interconnection networks has become an important issue and has been extensively studied (see, for example, [1,2,5–8,10–12,15–18]). Fault tolerance of interconnection networks is usually measured by how much of the network structure is preserved in the presence of a given number of component failures. Obviously, in the presence of component failures, the complete interconnection network is not available. Under this consideration, Becker and Simon [2] investigated a problem of what is the maximum number of dimensions that would be lost if the network contained a given number of faulty processors or links. They studied $f_H(n, k)$, the minimum number of faults, necessary for an adversary to destroy each $(n-k)$-dimensional subcube in an n-dimensional hypercube. Latifi [10] proposed a similar natural question of how large a part of a subnetwork, a smaller network but with the same topological properties as the original one, is still available in the network in the presence of component failures. He presented a bound on $F_S(n, k)$, the number of faulty nodes to make every $(n-k)$-dimensional substar faulty in an n-dimensional star graph and also determined the exact value of $F_S(n, k)$ when n is prime and $k = 2$ or when $n - 2 \leq k \leq n$. Wang and Yang [17] studied $F_B(n, k)$, the minimum number of faulty nodes to make every $(n-k)$-dimensional sub-bubble-sort graph faulty in an n-dimensional bubble-sort graph. They determined the exact value of $F_B(n, k)$ for some special cases and gave the lower and upper bounds on $F_B(n, k)$.

The interconnection network considered in this paper is the k-ary n-cube, denoted by Q_n^k, which has been proved to possess many attractive properties such as regularity, node transitivity and edge transitivity. Moreover, many interconnection networks can be viewed as the subclasses of Q_n^k, including the cycle, the torus and the hypercube. A number

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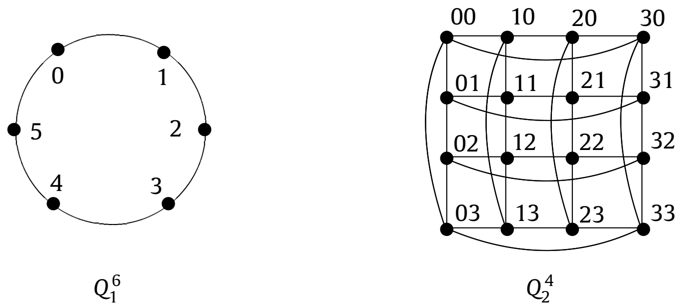

Fig. 1. Q_1^6 and Q_2^4.

of distributed memory multiprocessors have been built with a k-ary n-cube forming the underlying topology, such as the iWarp [14], the J-machine [13] and the Cray T3D [9]. In this paper, we are interested in the minimum number $f(n, m)$ of faulty nodes to make every $(n-m)$-dimensional subcube Q_{n-m}^k faulty in Q_n^k. We prove that $f(n, 0) = 1, f(n, 1) = k$ for odd $k \geq 3, f(n, n-1) = k^{n-1}$ for odd $k \geq 3$, and $k^m \leq f(n, m) \leq \binom{n-1}{m-1}k^m - \binom{n-2}{m-1}k^{m-1}$ for odd $k \geq 3$.

2. Preliminaries

In the remainder of this paper, we follow [3] for the graph-theoretical terminology and notation not defined here.

The k-ary n-cube Q_n^k ($k \geq 2$ and $n \geq 1$) is a graph consisting of k^n nodes, each of which has the form $u = u_{n-1}u_{n-2} \dots u_0$, where $0 \leq u_i \leq k-1$ for $0 \leq i \leq n-1$. Two nodes $u = u_{n-1}u_{n-2} \dots u_0$ and $v = v_{n-1}v_{n-2} \dots v_0$ are adjacent if and only if there exists an integer j, $0 \leq j \leq n-1$, such that $u_j = v_j \pm 1 \pmod k$ and $u_i = v_i$, for every $i \in \{0, 1, \dots, n-1\} \setminus \{j\}$. Such a link (u, v) is called a j-dimensional link. For clarity of presentation, we omit writing "(mod k)" in similar expressions for the remainder of the paper. Note that each node has degree $2n$ when $k \geq 3$, and n when $k = 2$. Obviously, Q_1^k is a cycle of length k, and Q_n^2 is an n-dimensional hypercube. We say that Q_n^k is divided into $Q_n^k[0], Q_n^k[1], \dots, Q_n^k[k-1]$ (abbreviated as $Q[0], Q[1], \dots, Q[k-1]$, if there are no ambiguities) along dimension d for some $0 \leq d \leq n-1$, where $Q[p]$, for every $0 \leq p \leq k-1$, is a subgraph of Q_n^k induced by $\{u = u_{n-1}u_{n-2} \dots u_d \dots u_0 \in V(Q_n^k) : u_d = p\}$. It is clear that each $Q[p]$ is isomorphic to Q_{n-1}^k for $0 \leq p \leq k-1$. Q_1^6 and Q_2^4 are shown in Fig. 1.

Let G and H be two graphs. G and H are distinct if their node sets are different, and disjoint if they have no common node. The Cartesian product of G and H, denoted by $G \times H$, is defined as follows: $V(G \times H) = V(G) \times V(H)$, two nodes u_1u_0 and v_1v_0 are adjacent in $G \times H$ if and only if $(u_1, v_1) \in E(G)$ and $u_0 = v_0$ or $(u_0, v_0) \in E(H)$ and $u_1 = v_1$. Let C_k be a cycle of length k. Then the Cartesian product of n C_k's $C_k \times C_k \times \cdots \times C_k$ and Q_n^k are obviously isomorphic. For two sets of nodes X and Y of G, denote by $[X, Y]$ the set of links with one end in X and the other end in Y. Let N_{k-1} be the set $\{0, 1, 2, \dots, k-1\}$ for an arbitrary integer $k \geq 2$.

Given two integers $n \geq 1$ and $k \geq 2$, for any integer m ($0 \leq m \leq n-1$), let $i_1, i_2, \dots, i_m$ be m integers with $0 \leq i_m < i_{m-1} < \cdots < i_1 \leq n-1$ and let $a_{i_1}, a_{i_2}, \dots, a_{i_m} \in N_{k-1}$. Denote $M = \{b_{n-1}b_{n-2} \dots b_{i_1+1}a_{i_1}b_{i_1-1}b_{i_1-2} \dots b_{i_2+1}a_{i_2} \dots a_{i_m}b_{i_m-1} b_{i_m-2} \dots b_0 : b_{n-1}, b_{n-2}, \dots, b_{i_1+1}, b_{i_1-1}, b_{i_1-2}, \dots, b_{i_2+1}, \dots, b_{i_m-1}, b_{i_m-2}, \dots, b_0 \in N_{k-1}\}$. In particular, $b_{n-1}b_{n-2} \dots b_{i_1+1}$ and $b_{i_m-1}b_{i_m-2} \dots b_0$ are empty strings if $i_1 = n-1$ and $i_m = 0$, respectively. Obviously, the subgraph of Q_n^k induced by M is isomorphic to Q_{n-m}^k. Let X be a *don't care* symbol and let $X^t = \underbrace{XX\dots X}_{t}$. For convenience of representation, we denote by an n-length string of symbols $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2} \dots a_{i_m}X^{i_m}$ the subgraph induced by M in Q_n^k. For example, $X^2 02$ in Q_4^3 denote the Q_2^3 induced by $\{0002, 0102, 0202, 1002, 1102, 1202, 2002, 2102, 2202\}$. In particular, when $m = 0$, $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2} \dots a_{i_m}X^{i_m}$ is just X^n, i.e., Q_n^k.

In addition, we can obtain the following lemma.

Lemma 1. *A Q_{n-m}^k in Q_n^k can be uniquely denoted by $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2} \dots a_{i_m}X^{i_m}$ for odd $k \geq 3$.*

Proof. We first prove the following two claims.

Claim 1. *Let C be a cycle of length k in Q_n^k. Then there exists $i \in \{0, 1, \dots, n-1\}$ such that C contains only i-dimensional links for odd $k \geq 3$.*

By contradiction. Suppose that C contains $i_1, i_2, \dots, i_s$-dimensional links, where $2 \leq s \leq n$ and $i_1, i_2, \dots, i_s \in \{0, 1, \dots, n-1\}$. For any $i_t \in \{i_1, i_2, \dots, i_s\}$, we divide Q_n^k into $Q[0], Q[1], \dots, Q[k-1]$ along dimension i_t. If $[V(Q[i]), V(Q[i+1])] \cap E(C) \neq \emptyset$ for every $i = 0, 1, \dots, k-1$, then there exist at least k distinct i_t-dimensional links in C. For $i_l \in \{i_1, i_2, \dots, i_s\}$ and $i_l \neq i_t$, since C contains i_l-dimensional links, we have $|E(C)| > k$, a contradiction. Hence, there exists at least one element $i^* \in N_{k-1}$ such that $[V(Q[i^*]), V(Q[i^*+1])] \cap E(C) = \emptyset$, say $i^* = k-1$. Note that $|[V(Q[i]), V(Q[i+1])] \cap E(C)|$ must be even for every $i = 0, 1, \ldots, k-2$. So, the number of i_t-dimensional links in C is even. Furthermore, by the arbitrariness of i_t, $|E(C)|$ is even, contrary to the fact that k is odd. The proof of Claim 1 is complete.

Claim 2. *For any Q_s^k $(2 \le s \le n-1)$ in Q_n^k, there exists pairwise distinct $j_1, j_2, \ldots, j_s \in \{0, 1, \ldots, n-1\}$ such that Q_s^k contains only $j_1, j_2, \ldots, j_s$-dimensional links for odd $k \ge 3$.*

Let $C = (0, 1, \ldots, k-1, 0)$ be a cycle of length k. Denote the Cartesian product of s C's $C \times \cdots \times C$ by H^*. For any two distinct nodes $u = u_{s-1}u_{s-2}\ldots u_0$ and $v = v_{s-1}v_{s-2}\ldots v_0$ in $V(H^*)$, u and v are joined with a j-dimensional link if and only if there exists an integer $j \in \{0, 1, \ldots, s-1\}$ such that $(u_j, v_j) \in E(C)$ and $u_l = v_l$ for every $l \in \{0, 1, \ldots, s-1\} \setminus \{j\}$. For $i = 0, 1, \ldots, s-1$, let C_i be a cycle of length k in H^*, which contains only i-dimensional links, such that the node $00\ldots0 \in V(C_i)$. Now, for any $i \in \{0, 1, \ldots, s-1\}$, if H^* contains i-dimensional links, then there exists C_i such that C_i contains i-dimensional links. Note that H^* and Q_s^k are isomorphic. So there exist s pairwise distinct cycles $H_1, H_2, \ldots, H_s$ of length k in Q_s^k such that if Q_s^k contains i-dimensional links, then there exists an integer $j \in \{1, 2, \ldots, s\}$ such that H_j contains i-dimensional links. By Claim 1, there exists $j_i \in \{0, 1, \ldots, n-1\}$ such that H_i contains only j_i-dimensional links for every $i = 1, 2, \ldots, s$. Hence, Q_s^k contains only $j_1, j_2, \ldots, j_s$-dimensional links. By the definition of Q_s^k, there exists pairwise distinct $i_1, i_2, \ldots, i_s \in \{0, 1, \ldots, n-1\}$ such that Q_s^k contains $i_1, i_2, \ldots, i_s$-dimensional links. So $\{j_1, j_2, \ldots, j_s\} = \{i_1, i_2, \ldots, i_s\}$. The proof of Claim 2 is complete.

Next, we prove Lemma 1 by induction on m. When $m = 0$, Q_n^k can be uniquely denoted by X^n. Assume that Lemma 1 is true for m, where $m \ge 0$. We shall show that Lemma 1 holds for $m+1$. Note that a $Q_{n-(m+1)}^k$ in Q_n^k must be in some Q_{n-m}^k. By the induction hypothesis, the Q_{n-m}^k can be uniquely denoted by $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2}\ldots a_{i_m}X^{i_m}$. By Claim 2, there exists pairwise distinct $j_1, j_2, \ldots, j_{n-m-1}, j_{n-m} \in \{0, 1, \ldots, n-1\} \setminus \{i_1, i_2, \ldots, i_m\}$ such that $Q_{n-(m+1)}^k$ contains only $j_1, j_2, \ldots, j_{n-m-1}$-dimensional links and Q_{n-m}^k contains only $j_1, j_2, \ldots, j_{n-m}$-dimensional links. The $Q_{n-(m+1)}^k$ can be obtained by dividing $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2}\ldots a_{i_m}X^{i_m}$ along dimension j_{n-m}. So there exists $a \in N_{k-1}$ such that, for any node $u_{n-1}u_{n-2}\ldots u_0 \in V(Q_{n-(m+1)}^k)$, $u_{j_{n-m}} = a$. Let $\{t_1, t_2, \ldots, t_{m+1}\} = \{i_1, \ldots, i_m, j_{n-m}\}$ with $t_1 > t_2 > \cdots > t_{m+1}$. Thus the $Q_{n-(m+1)}^k$ can be uniquely denoted by $X^{n-1-t_1}a_{t_1}X^{t_1-t_2-1}a_{t_2}\ldots a_{t_{m+1}}X^{t_{m+1}}$. The proof of Lemma 1 is complete. □

Lemma 2. *There are k^m disjoint Q_{n-m}^k's and $k^m\binom{n}{m}$ distinct Q_{n-m}^k's in Q_n^k for odd $k \ge 3$.*

Proof. This lemma is trivial when $m = 0$. In the following, we consider the case $m \ge 1$.

For odd $k \ge 3$, by Lemma 1, a Q_{n-m}^k in Q_n^k can be uniquely denoted by $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2}\ldots a_{i_m}X^{i_m}$, where $i_1, i_2, \ldots, i_m$ are m integers with $0 \le i_m < i_{m-1} < \cdots < i_1 \le n-1$ and $a_{i_1}, a_{i_2}, \ldots, a_{i_m} \in N_{k-1}$. According to the values of $i_1, i_2, \ldots, i_m$, we divide all the distinct Q_{n-m}^k's in Q_n^k into $\binom{n}{m}$ sets $A_1, A_2, \ldots, A_{\binom{n}{m}}$, where, for every $i \in \{1, 2, \ldots, \binom{n}{m}\}$, $A_i = \{X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2}\ldots a_{i_m}X^{i_m} : a_{i_1}, a_{i_2}, \ldots, a_{i_m} \in N_{k-1}\}$. Note that any two distinct Q_{n-m}^k's $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2}\ldots a_{i_m}X^{i_m}$ and $X^{n-1-i_1}b_{i_1}X^{i_1-i_2-1}b_{i_2}\ldots b_{i_m}X^{i_m}$ in A_i have no common node because there is some $l \in \{i_1, i_2, \ldots, i_m\}$ such that $a_l \ne b_l$. So, $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2}\ldots a_{i_m}X^{i_m}$ and $X^{n-1-i_1}b_{i_1}X^{i_1-i_2-1}b_{i_2}\ldots b_{i_m}X^{i_m}$ are disjoint. It follows that there are $|A_i| = k^m$ disjoint Q_{n-m}^k's in Q_n^k. For two distinct integers $i, j \in \{1, 2, \ldots, \binom{n}{m}\}$, any two Q_{n-m}'s $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2}\ldots a_{i_m}X^{i_m}$ in A_i and $X^{n-1-j_1}a_{j_1}X^{j_1-j_2-1}a_{j_2}\ldots a_{j_m}X^{j_m}$ in A_j are distinct, since otherwise the node set of $X^{n-1-i_1}a_{i_1}X^{i_1-i_2-1}a_{i_2}\ldots a_{i_m}X^{i_m}$ and the node set of $X^{n-1-j_1}a_{j_1}X^{j_1-j_2-1}a_{j_2}\ldots a_{j_m}X^{j_m}$ are the same, which yields that $i_1 = j_1, i_2 = j_2, \ldots, i_m = j_m$, a contradiction. Therefore, there are $\sum_{i=1}^{\binom{n}{m}} |A_i| = k^m\binom{n}{m}$ distinct Q_{n-m}^k's in Q_n^k. The proof is complete. □

3. Enumeration of faulty nodes

3.1. The lower and upper bounds

Given two integers $n \ge 1$ and $k \ge 2$, we are interested in finding $f(n, m)$, the minimum number of faulty nodes to make every $(n-m)$-dimensional subcube Q_{n-m}^k faulty in Q_n^k, where $0 \le m \le n-1$.

Lemma 3. *$k^m \le f(n, m) \le k^m\binom{n}{m}$ for odd $k \ge 3$.*

Proof. By Lemma 2, Q_n^k can be divided into k^m disjoint Q_{n-m}^k's. To damage all the disjoint Q_{n-m}^k's in Q_n^k, we need at least one faulty node for each Q_{n-m}^k, which yields that $f(n, m) \ge k^m$.

The upper bound on $f(n, m)$ can be obtained by making a node faulty in each of the $k^m\binom{n}{m}$ distinct Q_{n-m}^k's in Q_n^k. This will render: $f(n, m) \le k^m\binom{n}{m}$. Combining this with the fact that $f(n, m) \ge k^m$, the lemma follows. □

The following theorem gives the exact value of $f(n, m)$ for some special cases.

Theorem 1. *Let Q_n^k be a k-ary n-cube. Then the following hold.*

(1) $f(n, 0) = 1$.

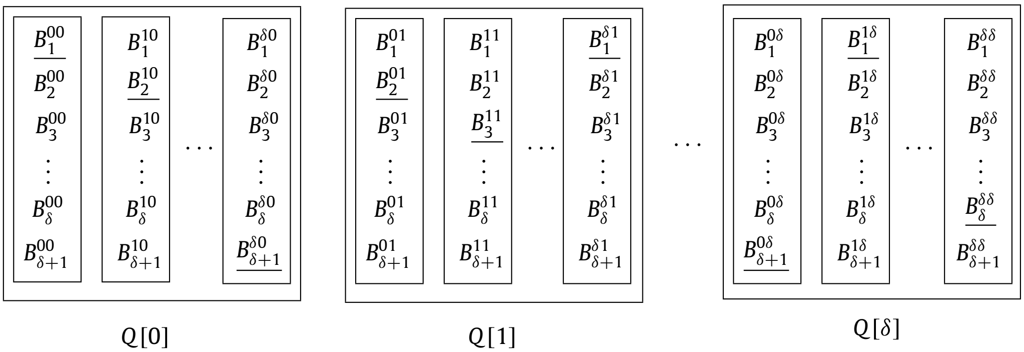

Fig. 2. The partition of $V(Q_n^k)$ and the selection of faulty nodes which are underlined (for convenience, denote $\delta = k - 1$).

(2) $f(n, 1) = k$ *for odd* $k \geq 3$.

(3) $f(n, n-1) = k^{n-1}$ *for odd* $k \geq 3$.

Proof. (1) Since the failure of a single node will damage the Q_n^k, we have $f(n, 0) \leq 1$. Lemma 3 implies that $f(n, 0) \geq k^0 = 1$. So $f(n, 0) = 1$.

(2) By Lemma 2, there are nk distinct Q_{n-1}^k's in Q_n^k. Note that, for every $i \in N_{k-1}$, the node $ii \dots i$ will damage the n Q_{n-1}^k's $iX^{n-1}, XiX^{n-2}, XXiX^{n-3}, \dots, X^{n-2}iX$ and $X^{n-1}i$. Therefore, the nodes $00\dots0, 11\dots1, \dots, (k-1)(k-1)\dots(k-1)$ will damage every Q_{n-1}^k in Q_n^k, which yields that $f(n, 1) \leq k$. Lemma 3 implies that $f(n, 0) \geq k$. So $f(n, 1) = k$.

(3) In the following, we consider the Q_n^k for odd $k \geq 3$. If $n = 1$, then, by (1), $f(n, n-1) = f(1, 0) = 1 = k^{1-1}$. Next, assume that $n \geq 2$. We divide Q_n^k into $Q[0], Q[1], \dots, Q[k-1]$ along dimension 0, where $Q[p]$, for every $0 \leq p \leq k-1$, is isomorphic to Q_{n-1}^k. Let $u^p = u_{n-1}u_{n-2}\dots u_1 p$ be a node in $Q[p]$, then the counterpart node of u^p in $Q[q]$ is denoted by u^q, where $u^q = u_{n-1}u_{n-2}\dots u_1 q$. Let $S^p \subseteq V(Q[p])$, then the counterpart node set of S^p in $Q[q]$ is denoted by S^q, where $S^q = \{x^q : x^p \in S^p\}$. Next, we prove a claim.

Claim. *There exists a partition* $A_1^0, A_2^0, \dots, A_k^0$ *of* $V(Q[0])$ *such that* $A_1^0 \cup A_2^1 \cup \dots \cup A_k^{k-1}$ *is the set of faulty nodes that damage all the* Q_1^k*'s in* Q_n^k*, where* $|A_i^0| = k^{n-2}$ *for every* $1 \leq i \leq k$*,* $A_i^0 \cap A_j^0 = \emptyset$ *for* $1 \leq i, j \leq k$ *and* $i \neq j$*,* $\bigcup_{i=1}^k A_i^0 = V(Q[0])$*, and* A_i^p *is the counterpart node set of* A_i^0 *in* $Q[p]$ *for* $1 \leq p \leq k-1$ *and* $1 \leq i \leq k$.

We prove the claim by induction on n. When $n = 2$, let $A_1^0 = \{00\}, A_2^0 = \{10\}, \dots, A_k^0 = \{(k-1)0\}$. Then $A_2^1 = \{11\}$, $A_3^2 = \{22\}, \dots, A_k^{k-1} = \{(k-1)(k-1)\}$. Clearly, $A_1^0 \cup A_2^1 \cup \dots \cup A_k^{k-1}$ is the set of faulty nodes that damage all the Q_1^k's in Q_2^k for odd $k \geq 3$. Assume that the claim is true for $n-1$, where $n \geq 3$. We shall show that the claim holds for n. Since $Q[0]$ is isomorphic to Q_{n-1}^k, we divide $Q[0]$ into $Q'[0], Q'[1], \dots, Q'[k-1]$ along dimension 1, where $Q'[p]$, for every $0 \leq p \leq k-1$, is induced by $\{u = u_{n-1}\dots u_1u_0 \in V(Q_n^k) : u_1 = p, u_0 = 0\}$. It is clear that each $Q'[p]$ is isomorphic to Q_{n-2}^k for $0 \leq p \leq k-1$. By the induction hypothesis, there exists a partition $B_1^{00}, B_2^{00}, \dots, B_k^{00}$ of $V(Q'[0])$ such that $B_1^{00} \cup B_2^{10} \cup \dots \cup B_k^{(k-1)0}$ is the set of faulty nodes that damage all the Q_1^k's in $Q[0]$, where $|B_i^{00}| = k^{n-3}$ for every $1 \leq i \leq k$, $B_i^{00} \cap B_j^{00} = \emptyset$ for $1 \leq i, j \leq k$ and $i \neq j$, $\bigcup_{i=1}^k B_i^{00} = V(Q'[0])$, and B_i^{p0} is the counterpart node set of B_i^{00} in $Q'[p]$ for $1 \leq p \leq k-1$ and $1 \leq i \leq k$. Denote the counterpart node set of B_i^{q0} in $Q[p]$ by B_i^{qp} for $1 \leq i \leq k$, $0 \leq q \leq k-1$ and $1 \leq p \leq k-1$. See Fig. 2 for more details about the partition.

Let $A_1^0 = B_1^{00} \cup B_2^{10} \cup \dots \cup B_k^{(k-1)0}$, $A_2^0 = B_2^{00} \cup B_3^{10} \cup \dots \cup B_k^{(k-2)0} \cup B_1^{(k-1)0}, \dots, A_k^0 = B_k^{00} \cup B_1^{10} \cup \dots \cup B_{k-1}^{(k-1)0}$. Then $A_2^1 = B_2^{01} \cup B_3^{11} \cup \dots \cup B_k^{(k-2)1} \cup B_1^{(k-1)1}, \dots, A_k^{k-1} = B_k^{0(k-1)} \cup B_1^{1(k-1)} \cup \dots \cup B_{k-1}^{(k-1)(k-1)}$. By Claim 1 in Lemma 1, Q_1^k in Q_n^k contains only i-dimensional links for some $i \in \{0, 1, \dots, n-1\}$. Clearly, all the Q_1^k's formed by 0-dimensional links are damaged by the faulty nodes in $A_1^0 \cup A_2^1 \cup \dots \cup A_k^{k-1}$ (see Fig. 2). Next, we show that all the Q_1^k's in $Q[1]$ are damaged by the faulty nodes in A_2^1. Define a mapping Ψ as follows:

$$\Psi : V(Q[0]) \to V(Q[1])$$
$$u_{n-1}\dots u_2u_1u_0 \mapsto u_{n-1}\dots u_2(u_1 - 1)(\bmod k)(u_0 + 1).$$

Ψ is an isomorphism between $Q[0]$ and $Q[1]$. For $S \subseteq V(Q[0])$, denote $\Psi(S) = \bigcup_{u \in S}\{\Psi(u)\}$. Then $\Psi(B_1^{00}) = B_1^{(k-1)1}$, $\Psi(B_2^{10}) = B_2^{01}, \dots, \Psi(B_k^{(k-1)0}) = B_k^{(k-2)1}$. Since all the Q_1^k's in $Q[0]$ are damaged by the faulty nodes in $A_1^0 = B_1^{00} \cup B_2^{10} \cup \dots \cup B_k^{(k-1)0}$, we have that all the Q_1^k's in $Q[1]$ are damaged by the faulty nodes in $\Psi(A_1^0) = A_2^1 = B_2^{01} \cup B_3^{11} \cup \dots \cup B_k^{(k-2)1} \cup B_1^{(k-1)1}$.

Table 1
The faulty nodes and damaged Q^k_{n-2}'s in A_{n-1} (for convenience, denote $\delta = k-1$).

Faulty nodes	Damaged Q^k_{n-2}'s
$00\ldots0$	$00X^{n-2},\ 0X0X^{n-3},\ \ldots,\ 0X^{n-2}0$
$01\ldots1$	$01X^{n-2},\ 0X1X^{n-3},\ \ldots,\ 0X^{n-2}1$
$\vdots$	$\vdots \quad \vdots \quad \ldots \quad \vdots$
$0\delta\ldots\delta$	$0\delta X^{n-2},\ 0X\delta X^{n-3},\ \ldots,\ 0X^{n-2}\delta$
$10\ldots0$	$10X^{n-2},\ 1X0X^{n-3},\ \ldots,\ 1X^{n-2}0$
$11\ldots1$	$11X^{n-2},\ 1X1X^{n-3},\ \ldots,\ 1X^{n-2}1$
$\vdots$	$\vdots \quad \vdots \quad \ldots \quad \vdots$
$1\delta\ldots\delta$	$1\delta X^{n-2},\ 1X\delta X^{n-3},\ \ldots,\ 1X^{n-2}\delta$
$\vdots$	$\vdots \quad \vdots \quad \ldots \quad \vdots$
$\delta0\ldots0$	$\delta0X^{n-2},\ \delta X0X^{n-3},\ \ldots,\ \delta X^{n-2}0$
$\delta1\ldots1$	$\delta1X^{n-2},\ \delta X1X^{n-3},\ \ldots,\ \delta X^{n-2}1$
$\vdots$	$\vdots \quad \vdots \quad \ldots \quad \vdots$
$\delta\delta\ldots\delta$	$\delta\delta X^{n-2},\ \delta X\delta X^{n-3},\ \ldots,\ \delta X^{n-2}\delta$

For every $2 \le p \le k-1$, define the mapping Ψ_p as follows:

$$\Psi_p: V(Q[0]) \to V(Q[p])$$
$$u_{n-1}\ldots u_2u_1u_0 \mapsto u_{n-1}\ldots u_2(u_1-p)\,(\mathrm{mod}\ k)\,(u_0+p).$$

Similarly, we have that all the Q^k_1's in $Q[p]$ are damaged by the faulty nodes in A^p_{p+1} for every $2 \le p \le k-1$. Therefore $A^0_1 \cup A^1_2 \cup \cdots \cup A^{k-1}_k$ is the set of faulty nodes that damage all the Q^k_1's in Q^k_n. The proof of the claim is complete.

By the claim, $f(n, n-1) \le |A^0_1|+|A^1_2|+\cdots+|A^{k-1}_k| = kk^{n-2} = k^{n-1}$. Lemma 3 implies that $f(n, n-1) \ge k^{n-1}$. Therefore we have $f(n, n-1) = k^{n-1}$. □

Note that for some special cases the exact value of $f(n, m)$ coincides with the lower bound. But there is a large gap between the lower bound k^m and the upper bound $k^m\binom{n}{m}$ in Lemma 3. In the following, we shall improve the upper bound on $f(n, m)$.

3.2. A better upper bound on $f(n, m)$

We first present a better upper bound on $f(n, 2)$ by giving a set of faulty nodes that damage all Q^k_{n-2}'s in Q^k_n.

Lemma 4. *Denote by $f(n, 2)$ the minimum number of faulty nodes that make every $(n-2)$-dimensional subcube Q^k_{n-2} faulty in Q^k_n. Then $f(n, 2) \le \binom{n-1}{1}k^2 - \binom{n-2}{1}k$ for odd $k \ge 3$.*

Proof. By Lemma 2, there are $\binom{n}{2}k^2$ distinct Q^k_{n-2}'s in Q^k_n. Divide all the distinct Q^k_{n-2}'s in Q^k_n into $n-1$ sets $A_{n-1}, A_{n-2}, \ldots, A_1$, where, for $i \in \{1, 2, \ldots, n-1\}$, $A_i = \{X^{n-1-i}a_iX^{i-j-1}a_jX^j : 0 \le j \le i-1 \text{ and } a_i, a_j \in N_{k-1}\}$. Clearly, $|A_i| = \binom{i}{1}k^2$ for $i \in \{1, 2, \ldots, n-1\}$. We first find k^2 faulty nodes to damage all the Q^k_{n-2}'s in A_{n-1}. See Table 1 for more details.

Secondly, we find k^2 faulty nodes to damage all the Q^k_{n-2}'s in A_{n-2}. See Table 2 for more details.

We proceed in a similar way until we find k^2 faulty nodes to damage all the Q^k_{n-2}'s in A_1. See Table 3 for more details of A_1.

Note that the nodes $00\ldots0, 11\ldots1, \ldots, (k-1)(k-1)\ldots(k-1)$ repeat $n-2$ times in the faulty nodes which we found. Except $00\ldots0, 11\ldots1, \ldots, (k-1)(k-1)\ldots(k-1)$, the faulty nodes are pairwise distinct. Thus we found $(n-1)k^2-(n-2)k$ faulty nodes. Since $|A_{n-1}|+|A_{n-2}|+\cdots+|A_1| = (n-1)k^2+(n-2)k^2+\cdots+k^2 = \binom{n}{2}k^2$, the faulty nodes which we found can damage all the Q^k_{n-2}'s in Q^k_n, which yields $f(n, 2) \le (n-1)k^2-(n-2)k = \binom{n-1}{1}k^2 - \binom{n-2}{1}k$. The proof is complete. □

More generally, we give a better upper bound on $f(n, m)$. The following lemma is useful.

Lemma 5 ([4]). *Let s, t be two nonnegative integers with $s \ge t$. Then $\binom{s-1}{t-1} + \binom{s-2}{t-1} + \cdots + \binom{t-1}{t-1} = \binom{s}{t}$*

Theorem 2. *Denote by $f(n, m)$ the minimum number of faulty nodes that make every $(n-m)$-dimensional subcube Q^k_{n-m} faulty in Q^k_n. Then $f(n, m) \le \binom{n-1}{m-1}k^m - \binom{n-2}{m-1}k^{m-1}$ for odd $k \ge 3$.*

Table 2
The faulty nodes and damaged Q^k_{n-2}'s in A_{n-2} (for convenience, denote $\delta = k-1$).

Faulty nodes	Damaged Q^k_{n-2}'s
$000\ldots0$	$X00X^{n-3}$, $X0X0X^{n-4}$, $\ldots$, $X0X^{n-3}0$
$001\ldots1$	$X01X^{n-3}$, $X0X1X^{n-4}$, $\ldots$, $X0X^{n-3}1$
$\vdots$	$\vdots$ $\vdots$ $\ldots$ $\vdots$
$00\delta\ldots\delta$	$X0\delta X^{n-3}$, $X0X\delta X^{n-4}$, $\ldots$, $X0X^{n-3}\delta$
$110\ldots0$	$X10X^{n-3}$, $X1X0X^{n-4}$, $\ldots$, $X1X^{n-3}0$
$111\ldots1$	$X11X^{n-3}$, $X1X1X^{n-4}$, $\ldots$, $X1X^{n-3}1$
$\vdots$	$\vdots$ $\vdots$ $\ldots$ $\vdots$
$11\delta\ldots\delta$	$X1\delta X^{n-3}$, $X1X\delta X^{n-4}$, $\ldots$, $X1X^{n-3}\delta$
$\vdots$	$\vdots$ $\vdots$ $\ldots$ $\vdots$
$\delta\delta0\ldots0$	$X\delta0X^{n-3}$, $X\delta X0X^{n-4}$, $\ldots$, $X\delta X^{n-3}0$
$\delta\delta1\ldots1$	$X\delta1X^{n-3}$, $X\delta X1X^{n-4}$, $\ldots$, $X\delta X^{n-3}1$
$\vdots$	$\vdots$ $\vdots$ $\ldots$ $\vdots$
$\delta\delta\delta\ldots\delta$	$X\delta\delta X^{n-3}$, $X\delta X\delta X^{n-4}$, $\ldots$, $X\delta X^{n-3}\delta$

Table 3
The faulty nodes and damaged Q^k_{n-2}'s in A_1 (for convenience, denote $\delta = k-1$).

Faulty nodes	Damaged Q^k_{n-2}'s
$0\ldots00$	$X^{n-2}00$
$0\ldots01$	$X^{n-2}01$
$\vdots$	$\vdots$
$0\ldots0\delta$	$X^{n-2}0\delta$
$1\ldots10$	$X^{n-2}10$
$1\ldots11$	$X^{n-2}11$
$\vdots$	$\vdots$
$1\ldots1\delta$	$X^{n-2}1\delta$
$\ldots$	$\ldots$
$\delta\ldots\delta0$	$X^{n-2}\delta0$
$\delta\ldots\delta1$	$X^{n-2}\delta1$
$\vdots$	$\vdots$
$\delta\ldots\delta\delta$	$X^{n-2}\delta\delta$

Proof. We prove the theorem by induction on m. By Lemma 4, the theorem holds for $m=2$. Assume the theorem holds for $m-1$ ($m\geq 3$), i.e., $f(n,m-1)\leq \binom{n-1}{m-2}k^{m-1}-\binom{n-2}{m-2}k^{m-2}$ for odd $k\geq 3$. We shall show that the theorem holds for m. By Lemma 2, there are $\binom{n}{m}k^m$ distinct Q^k_{n-m}'s in Q^k_n. Divide all the distinct Q^k_{n-m}'s in Q^k_n into $n-m+1$ sets $A_{n-1}, A_{n-2}, \ldots, A_{m-1}$, where, for every $i\in\{m-1,\ldots,n-2,n-1\}$, $A_i=\{X^{n-1-i}a_iX^{i-i_2-1}a_{i_2}X^{i_2-i_3-1}a_{i_3}\ldots a_{i_m}X^{i_m} : 0\leq i_m<\cdots<i_3<i_2\leq i-1 \text{ and } a_i, a_{i_2}, a_{i_3}, \ldots, a_{i_m}\in N_{k-1}\}$ and $|A_i|=\binom{i}{m-1}k^m$. By Lemma 5, $|A_{n-1}|+|A_{n-2}|+\cdots+|A_{m-1}|=(\binom{n-1}{m-1}+\binom{n-2}{m-1}+\cdots+\binom{m-1}{m-1})k^m=\binom{n}{m}k^m$. For every $i\in\{m-1,m,\ldots,n-1\}$, denote by B_i the set of faulty nodes with the minimum cardinality to damage all Q^k_{n-m}'s in A_i. For some $r\in N_{k-1}$ and every $i\in\{m-1,m,\ldots,n-1\}$, we denote by $A_i(r)$ the set $\{X^{n-1-i}rX^{i-i_2-1}a_{i_2}X^{i_2-i_3-1}a_{i_3}\ldots a_{i_m}X^{i_m} : 0\leq i_m<\cdots<i_3<i_2\leq i-1 \text{ and } a_{i_2}, a_{i_3}, \ldots, a_{i_m}\in N_{k-1}\}$, and denote by $B_i(r)$ the set of faulty nodes with the minimum cardinality to damage all Q^k_{n-m}'s in $A_i(r)$. Clearly, for every $i\in\{m-1,m,\ldots,n-1\}$, $\bigcup_{r=0}^{k-1}B_i(r)=B_i$ and $B_i(r)\cap B_i(r')=\emptyset$, where $r, r'\in N_{k-1}$ and $r\neq r'$.

First, we shall find $B_{n-1}(r)$ for every $r\in N_{k-1}$. Denote $A'_{n-1}=\{X^{n-i_2-2}a_{i_2}X^{i_2-i_3-1}a_{i_3}\ldots a_{i_m}X^{i_m} : 0\leq i_m<\cdots<i_3<i_2\leq n-2 \text{ and } a_{i_2}, a_{i_3}, \ldots, a_{i_m}\in N_{k-1}\}$. Note that A'_{n-1} is the set of all the $Q^k_{(n-1)-(m-1)}$'s in Q^k_{n-1}. By the induction hypothesis, there exists a set B'_{n-1} of faulty nodes to damage all the $Q^k_{(n-1)-(m-1)}$'s in Q^k_{n-1}

with $|B'_{n-1}| = f(n-1, m-1) \le \binom{n-2}{m-2}k^{m-1} - \binom{n-3}{m-2}k^{m-2}$. Thus the faulty nodes in B'_{n-1} damage all the $Q^k_{(n-1)-(m-1)}$'s in A'_{n-1}. Let $B^*_{n-1}(r) = \{ru_{n-2}u_{n-3}\dots u_0 \in V(Q^k_n) : u_{n-2}u_{n-3}\dots u_0 \in B'_{n-1}\}$ be a set of faulty nodes. Then $|B^*_{n-1}(r)| = |B'_{n-1}|$. Recall that $A_{n-1}(r) = \{rX^{n-i_2-2}a_{i_2}X^{i_2-i_3-1}a_{i_3}\dots a_{i_m}X^{i_m} : 0 \le i_m < \dots < i_3 < i_2 \le n-2 \text{ and } a_{i_2}, a_{i_3}, \dots, a_{i_m} \in N_{k-1}\}$. The faulty nodes in $B^*_{n-1}(r)$ damage all the Q^k_{n-m}'s in $A_{n-1}(r)$. Therefore $|B_{n-1}(r)| \le |B^*_{n-1}(r)| = |B'_{n-1}| = f(n-1, m-1) \le \binom{n-2}{m-2}k^{m-1} - \binom{n-3}{m-2}k^{m-2}$. Note that for any $r' \in N_{k-1} \setminus \{r\}$, $|B_{n-1}(r')| = |B_{n-1}(r)|$. It follows that $|B_{n-1}| = \sum_{r=0}^{k-1} |B_{n-1}(r)| \le k(\binom{n-2}{m-2}k^{m-1} - \binom{n-3}{m-2}k^{m-2}) = \binom{n-2}{m-2}k^m - \binom{n-3}{m-2}k^{m-1}$.

Next, we shall find $B_{n-2}(r)$ for every $r \in N_{k-1}$. Denote $A'_{n-2} = \{X^{n-i_2-3}a_{i_2}X^{i_2-i_3-1}a_{i_3}\dots a_{i_m}X^{i_m} : 0 \le i_m < \dots < i_3 < i_2 \le n-3 \text{ and } a_{i_2}, a_{i_3}, \dots, a_{i_m} \in N_{k-1}\}$. Note that A'_{n-2} is the set of all the $Q^k_{(n-2)-(m-1)}$'s in Q^k_{n-2}. By the induction hypothesis, there exists a set B'_{n-2} of faulty nodes to damage all the $Q^k_{(n-2)-(m-1)}$'s in Q^k_{n-2} with $|B'_{n-2}| = f(n-2, m-1) \le \binom{n-3}{m-2}k^{m-1} - \binom{n-4}{m-2}k^{m-2}$. Thus the faulty nodes in B'_{n-2} damage all the $Q^k_{(n-2)-(m-1)}$'s in A'_{n-2}. Given an integer $a \in N_{k-1}$, let $B^*_{n-2}(r) = \{aru_{n-3}u_{n-4}\dots u_0 \in V(Q^k_n) : u_{n-3}u_{n-4}\dots u_0 \in B'_{n-2}\}$ be a set of faulty nodes. Then $|B^*_{n-2}(r)| = |B'_{n-2}|$. Recall that $A_{n-2}(r) = \{XrX^{n-i_2-3}a_{i_2}X^{i_2-i_3-1}a_{i_3}\dots a_{i_m}X^{i_m} : 0 \le i_m < \dots < i_3 < i_2 \le n-3 \text{ and } a_{i_2}, a_{i_3}, \dots, a_{i_m} \in N_{k-1}\}$. The faulty nodes in $B^*_{n-2}(r)$ damage all the Q^k_{n-m}'s in $A_{n-2}(r)$. Therefore $|B_{n-2}(r)| \le |B^*_{n-2}(r)| = |B'_{n-2}| = f(n-2, m-1) \le \binom{n-3}{m-2}k^{m-1} - \binom{n-4}{m-2}k^{m-2}$. Note that for any $r' \in N_{k-1} \setminus \{r\}$, $|B_{n-2}(r')| = |B_{n-2}(r)|$. It follows that $|B_{n-2}| = \sum_{r=0}^{k-1} |B_{n-2}(r)| \le k(\binom{n-3}{m-2}k^{m-1} - \binom{n-4}{m-2}k^{m-2}) = \binom{n-3}{m-2}k^m - \binom{n-4}{m-2}k^{m-1}$.

We proceed in a similar way until we get $|B_m| \le \binom{m-1}{m-2}k^m - \binom{m-2}{m-2}k^{m-1}$.

Now, we consider the remaining $A_{m-1} = \{X^{n-m}a_{m-1}a_{m-2}\dots a_0 : a_0, \dots, a_{m-2}, a_{m-1} \in N_{k-1}\}$. There are k^m disjoint Q^k_{n-m}'s in A_{m-1}. By making a node faulty in each of the k^m disjoint Q^k_{n-m}'s in A_{m-1}, we have $|B_{m-1}| = k^m = \binom{m-2}{m-2}k^m$. Therefore,

$$\begin{aligned} f(n,m) &= |B_{n-1}| + |B_{n-2}| + \dots + |B_m| + |B_{m-1}| \\ &\le \binom{n-2}{m-2}k^m - \binom{n-3}{m-2}k^{m-1} + \binom{n-3}{m-2}k^m - \binom{n-4}{m-2}k^{m-1} \\ &\quad + \dots + \binom{m-1}{m-2}k^m - \binom{m-2}{m-2}k^{m-1} + \binom{m-2}{m-2}k^m \\ &= \left(\binom{n-2}{m-2} + \binom{n-3}{m-2} + \dots + \binom{m-2}{m-2}\right)k^m \\ &\quad - \left(\binom{n-3}{m-2} + \binom{n-4}{m-2} + \dots + \binom{m-2}{m-2}\right)k^{m-1}. \end{aligned}$$

By Lemma 5, $f(n,m) \le \binom{n-1}{m-1}k^m - \binom{n-2}{m-1}k^{m-1}$. The proof is complete. □

4. Conclusions

In this paper, we investigate $f(n,m)$, the minimum number of faulty nodes which make every $(n-m)$-dimensional subcube Q^k_{n-m} faulty in a k-ary n-cube Q^k_n under node-failure models. We present the lower and upper bounds on $f(n,m)$, and determine the exact value of $f(n,m)$ for some special cases. The results can be used in the reliability analysis of the subnetworks in k-ary n-cubes. The determination of the exact value of $f(n,m)$ remains an open problem for the general case.

References

[1] Yaagoub A. Ashir, Iain A. Stewart, Fault-tolerant embeddings of Hamiltonian circuits in k-ary n-cubes, SIAM Journal on Discrete Mathematics 15 (3) (2002) 317–328.
[2] Bernd Becker, Hans-Ulrich Simon, How robust is the n-cube? in: Proceedings of 27th Annual Symposium on Foundations of Computer Science, 1986, pp. 283–291.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[4] Richard A. Brualdi, Introductory Combinatorics, 3rd edition, Prentice Hall, New Jersey, 1999.
[5] Mee Yee Chan, Shiang-Jen Lee, On the existence of Hamiltonian circuits in faulty hypercubes, SIAM Journal on Discrete Mathematics 4 (4) (1991) 511–527.
[6] Jung-Sheng Fu, Fault-free Hamiltonian cycles in twisted cubes with conditional link faults, Theoretical Computer Science 407 (1–3) (2008) 318–329.
[7] Tung-Yang Ho, Ting-Yi Sung, Lih-Hsing Hsu, A note on edge fault tolerance with respect to hypercubes, Applied Mathematics Letters 18 (10) (2005) 1125–1128.
[8] Sun-Yuan Hsieh, Embedding longest fault-free paths onto star graphs with more vertex faults, Theoretical Computer Science 337 (1–3) (2005) 370–378.
[9] R.E. Kessler, J.L. Schwarzmeier, Cray T3D: a new dimension for Cray Research, in: Proceedings of the 38th IEEE Computer Society International Conference, Compcon Spring'93, San Francisco, CA, 1993, pp. 176–182.
[10] Shahram Latifi, A study of fault tolerance in star graph, Information Processing Letters 102 (5) (2007) 196–200.
[11] Shangwei Lin, Shiying Wang, Chunfang Li, Panconnectivity and edge-pancyclicity of k-ary n-cubes with faulty elements, Discrete Applied Mathematics 159 (4) (2011) 212–223.

[12] Meijie Ma, Guizhen Liu, Jun-Ming Xu, Fault-tolerant embedding of paths in crossed cubes, Theoretical Computer Science 407 (1–3) (2008) 110–116.
[13] Michael Noakes, William J. Dally, System design of the *J*-machine, in: Proceedings of the Sixth MIT Conference on Advanced Research in VLSI, MIT Press, Cambridge, MA, 1990, pp. 179–194.
[14] Craig Peterson, James Sutton, Paul Wiley, iWarp: a 100-MOPS, LIW microprocessor for multicomputers, IEEE Micro 11 (3) (1991) 26–29. 81–87.
[15] Iain A. Stewart, Yonghong Xiang, Embedding long paths in k-ary n-cubes with faulty nodes and links, IEEE Transactions on Parallel and Distributed Systems 19 (8) (2008) 1071–1085.
[16] Ping-Ying Tsai, Jung-Sheng Fu, Gen-Huey Chen, Fault-free longest paths in star networks with conditional link faults, Theoretical Computer Science 410 (8–10) (2009) 766–775.
[17] Shiying Wang, Yuxing Yang, Fault tolerance in bubble-sort graph networks, Theoretical Computer Science 421 (2012) 62–69.
[18] Ming-Chien Yang, Jimmy J.M. Tan, Lih-Hsing Hsu, Hamiltonian circuit and linear array embeddings in faulty k-ary n-cubes, Journal of Parallel and Distributed Computing 67 (4) (2007) 362–368.