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Pessimistic rough set based decisions: A multigranulation fusion strategy ☆

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ABSTRACT

Multigranulation rough sets (MGRS) is one of desirable directions in rough set theory, in which lower/upper approximations are approximated by granular structures induced by multiple binary relations. It provides a new perspective for decision making analysis based on rough set theory. In decision making analysis, people often adopt the decision strategy "Seeking common ground while eliminating differences" (SCED). This strategy implies that one reserves common decisions while deleting inconsistent decisions. From this point of view, the objective of this study is to develop a new multigranulation rough set based decision model based on SCED strategy, called pessimistic multigranulation rough sets. We study this model from three aspects, which are lower/upper approximation and their properties, decision rules and attribute reduction, in this paper.

1. Introduction

Rough set theory, originated by Pawlak [20,21], has become a well-established theory for uncertainty management in a wide variety of applications related to pattern recognition, image processing, feature selection, neural computing, conflict analysis, decision support, data mining and knowledge discovery [2,4,10,11,15,16,25–28,30,31,34,41,43]. One of the strengths of rough set theory is the fact that all its parameters are obtained from the given data. In other words, instead of using external numbers or other additional parameters, *the rough set data analysis* (RSDA) utilizes solely the granular structure of the given data, expressed as classes of suitable equivalence relations [5,6,25,26,33].

In the past ten years, several extensions of the rough set model have been proposed in terms of various requirements, such as the probabilistic rough set model (see [38]), the variable precision rough set (VPRS) model (see [42,45]), the rough set model based on tolerance relation (see [12–14]), the Bayesian rough set model (see [32]), the Dominance-based rough set model (see [3]), game-theoretic rough set model (see [7,8]), the fuzzy rough set model and the rough fuzzy set model (see [1,19]). In particular, the probabilistic rough sets have been paid close attention [9,35–37,39]. A special issue on probabilistic rough sets was set up in International Journal of Approximate Reasoning, in which six relative papers were published [36]. Yao presented a new decision making method based on the probabilistic rough set, called three-way decision, which are constructed by positive region, boundary region and negative region, respectively [39]. In the literature [37], the author

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further emphasized the superiority of three-way decisions in probabilistic rough set models. In fact, the probabilistic rough sets is developed based on the Bayesian decision principle, in which its parameters can be learned from a given decision table. Three-way decisions are most of superiorities of probabilistic rough set models. Since the fuzzy rough sets was proposed, this theory have been also largely developed [10,31,44]. Jensen and Shen [10] developed a series of feature selection approaches for classification using the fuzzy rough set model. Shen and Chouchoulas [31] presented a rough-fuzzy approach for generating classification rules. Zhao et al. [44] addressed a fuzzy variable rough set model and trained a rule-based classifier with it. Indeed, the fuzzy rough set theory can do better than other rough set models when we need to deal with numeric data, while the classical rough sets will collapse. In fact, the classical rough set theory will be dominant when all of the attribute values and concepts are discrete-valued.

In the view of granular computing (proposed by Zadeh [40]), in existing rough set models, a general concept described by a set is always characterized via the so-called upper and lower approximations under a single granulation, i.e., the concept is depicted by known knowledge induced from a single relation (such as equivalence relation, tolerance relation and reflexive relation) on the universe [17,18,22,38]. Conveniently, this kind of rough set models is called single granulation rough sets, just SGRS. However, this approach to describing a target concept is mainly based on the following assumption:

If P and Q are two sets of conditional features and $X \subseteq U$ is a target concept, then the rough sets of X are derived from the quotient set $U/(P \cup Q)$. In fact, the quotient set is equivalent to the formula

$$\widehat{P \cup Q} = \{P_i \cap Q_j : \ P_i \in U/P, \ Q_j \in U/Q, \ P_i \cap P_j \neq \emptyset\}.$$

It implies the following two ideas:

(1) we can perform an intersection operation between any P_i and Q_j,
(2) the target concept is approximately described by using the quotient set $U/(P \cup Q)$.

In fact, the target concept is described by using a finer granulation (partitions) formed through combining two known granulations (partitions) induced from two attribute subsets. Although it generates a much finer granulation, the combination/fining destroys the original granular structure/partitions. In general, the above assumption cannot be always satisfied or required in practice. In many circumstances, we often need to describe concurrently a target concept through multi binary relations (e.g. equivalence relation, tolerance relation, reflexive relation and neighborhood relation) on the universe according to a user's requirements or targets of problem solving. Based on this consideration, Qian et al. [23–25] introduced multigranulation rough set theory (MGRS) to more widely apply rough set theory in practical applications, in which lower/upper approximations are approximated by granular structures induced by multiple binary relations.

Besides the motivation of theoretical study above, it also has the motivation of real application. From the viewpoint of rough set's application, the multigranulation rough set theory is very desirable in many real applications, such as multi-source data analysis, knowledge discovery from data with high dimensions and distributive information systems. For example, It is very desirable to develop multigranulation rough sets in the following two cases.

(1) When we apply the rough set theory for data mining and knowledge discovery from multi-source data, its key task is to consider how to knowledge representation and rough approximation in the context of multi-source information systems. In order to efficiently discover knowledge online, it is unnecessary to gather and combine every information systems from multiple sources as an entire information system for data analysis. More reasonable strategy is to directly analyzing these multi-source information systems. In this situation, the classical single granulation rough set theory (SGRS) has its limitation that the computational times of algorithms are too longer to efficiently knowledge discovery from multi-source information systems.
(2) When analyzing data with high dimensions, a lot of attributes bring out a challenge for knowledge discovery. There are two main problems: (1) after granulating data using all attributes, the intensions of information granules obtained will be very longer and the extensions of those will be very smaller, which determines a rule-based classifier with much smaller generalization ability; and (2) a lot of attributes also lead to inefficient of algorithms in rough set theory. These two shortcomings are so important that the existing rough set models cannot be well used to rough set-based data analysis for data with high dimensions.

In the multigranulation rough set theory, each of various binary relation determines a corresponding information granulation, which largely impacts the commonality between each of the granulations and the fusion among all granulations. In this paper, we do not further discuss how binary relations impact information fusion among all granulations,[1] but develop a new decision method, called a pessimistic multigranulation rough set model.

Classical rough sets and multigranulation rough sets are complementary in many practical applications. When two attribute sets in information systems possesses a contradiction or inconsistent relationship, or efficient computation is required,

[1] How each of various binary relations impacts information fusion among information granulations is a very important and interesting issue in the multigranulation rough set theory. This will produce many information fusion methods, especially the information fusion in various kinds of granular spaces. This study is beyond the scope of this paper. We will investigate this issue in further work.

MGRS will display its advantage for rule extraction and knowledge discovery; when there is a consistent relationship between its values under one-attribute set and those under another attribute set, standard rough set theory (SGRS) will hold dominant position. In particular, for some practical applications in which the above two cases occur concurrently, these two methods can be combined to solve problems.

The first multigranulation rough set model was proposed by Qian et al. in Refs. [23,28] to deal with complete information systems, the second is developed for rough set based decision from incomplete data [24,25]. In these two models, each lower approximation collects the objects that one of their equivalence classes is included in the target concept in multiple granular structures. This kind of multigranulation rough sets is called optimistic multigranulation rough set. From the viewpoint of decision making, this kind of rough set based decision can be called optimistic rough set based decision, which is based on the "Seeking common ground while reserving differences" (SCRD) strategy.

In decision making field, people often adopt another decision strategy "Seeking common ground while eliminating differences" (SCED). This strategy implies that one reserves common decisions while deleting inconsistent decisions. Hence, this opinion can be seen as a conservative decision strategy. For example, this strategy often is adopted in many decision problems such as conflict analysis, selection decision and risk investment, and so on. As a powerful tool to deal with decision problems, rough set theory have contributed many important results. Hence, it is very desirable that one investigates rough set theory in the context of "Seeking common ground while eliminating differences" (SCED) strategy. In this study, our objective is to develop a new multigranulation rough decision theory based on the SCED strategy, called pessimistic multigranulation rough sets. Following these comments, multigranulation rough set theory (MGRS) can be classified into two parts: one is the optimistic multigranulation rough set [28] and the other is the pessimistic multigranulation rough set [27]. By the way, the words "optimistic" and "pessimistic" have also occurred in some rough set literatures [37,39], however they have different semantics. The words "optimistic" and "pessimistic" in this series of studies about multigranulation rough sets refer in particular to two information fusion strategies "Seeking common ground while reserving differences" and "Seeking common ground while eliminating differences", respectively, while these two words do not in other literatures.

The study is organized as follows. Some basic concepts in classical rough sets and multigranulation rough sets are briefly reviewed in Section 2. In Section 3, we propose a pessimistic rough set based decision method called pessimistic multigranulation rough sets, and investigate some of its nice properties. In Section 4, we focus on decision rules extracted from multiple granular structures using the proposed pessimistic rough set based decision method. In Section 5, we develop attribute reduction approaches in the context of pessimistic multigranulation rough sets, and give a heuristic approach to computing a reduct in the pessimistic multigranulation rough set model. In Section 6, A practical case is employed for illustrating the mechanism and application of the pessimistic multigranulation rough sets. Finally, Section 7 concludes this paper by bringing some remarks and discussions.

2. Preliminary knowledge on rough sets

In this section, we review some basic concepts such as information system, Pawlak's rough set, and optimistic multigranulation rough set. Throughout this paper, we assume that the universe U is a finite non-empty set.

2.1. Pawlak's rough set

Formally, an information system can be considered as a pair $I = \langle U, AT\rangle$, where.

- U is a non-empty finite set of objects, it is called the universe;
- AT is a non-empty finite set of attributes, such that $\forall a \in AT, V_a$ is the domain of attribute a.

$\forall x \in U$, we denote the value of x under the attribute a ($a \in AT$) by $a(x)$. Given $A \subseteq AT$, an indiscernibility relation $IND(A)$ can be defined as

$$IND(A) = \{(x,y) \in U \times U : \ a(x) = a(y),\ a \in A\}. \tag{1}$$

The relation $IND(A)$ is reflexive, symmetric and transitive, then $IND(A)$ is an equivalence relation. By the indiscernibility relation $IND(A)$, one can derive the lower and upper approximations of an arbitrary subset X of U. They are defined as

$$\underline{A}(X) = \{x \in U : \ [x]_A \subseteq X\} \text{ and } \overline{A}(X) = \{x \in U : \ [x]_A \cap X \neq \emptyset\} \tag{2}$$

respectively, where $[x]_A = \{y \in U : \ (x,y) \in IND(A)\}$ is the A-equivalence class containing x. The pair $[\underline{A}(X), \overline{A}(X)]$ is referred to as the Pawlak's rough set of X with respect to the set of attributes A.

2.2. Optimistic multigranulation rough set

The multigranulation rough set (MGRS) is different from Pawlak's rough set model because the former is constructed on the basis of a family of indiscernibility relations instead of single indiscernibility relation.

In optimistic multigranulation rough set approach, the word "optimistic" is used to express the idea that in multi independent granular structures, we need only at least one granular structure to satisfy with the inclusion condition between equivalence class and the approximated target. The upper approximation of optimistic multigranulation rough set is defined by the complement of the lower approximation.

Definition 1 [29]. Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, then $\forall X \subseteq U$, the optimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{i=1}^m A_i}^O(X)$ and $\overline{\sum_{i=1}^m A_i}^O(X)$, respectively,

$$\underline{\sum_{i=1}^{m} A_i}^O(X) = \{x \in U: [x]_{A_1} \subseteq X \vee [x]_{A_2} \subseteq X \vee \cdots \vee [x]_{A_m} \subseteq X\}; \tag{3}$$

$$\overline{\sum_{i=1}^{m} A_i}^O(X) = \sim \left(\underline{\sum_{i=1}^{m} A_i}^O(\sim X)\right); \tag{4}$$

where $[x]_{A_i}$ $(1 \leqslant i \leqslant m)$ is the equivalence class of x in terms of set of attributes A_i, and $\sim X$ is the complement of X.

By the lower approximation $\underline{\sum_{i=1}^m A_i}^O(X)$ and upper approximation $\overline{\sum_{i=1}^m A_i}^O(X)$, the optimistic multigranulation boundary region of X is

$$BN^O_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^O(X) - \underline{\sum_{i=1}^{m} A_i}^O(X). \tag{5}$$

Theorem 1. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, then $\forall X \subseteq U$, we have*

$$\overline{\sum_{i=1}^{m} A_i}^O(X) = \{x \in U: [x]_{A_1} \cap X \neq \emptyset \wedge [x]_{A_2} \cap X \neq \emptyset \wedge \cdots \wedge [x]_{A_m} \cap X \neq \emptyset\}. \tag{6}$$

From Theorem 1, it can be seen that though the optimistic multigranulation upper approximation is defined by the complement of the optimistic multigranulation lower approximation, it can also be considered as a set in which objects have non-empty intersection with the target in terms of each granular structure.

Since multiple independent indiscernibility relations are used in optimistic multigranulation rough set, then it is natrual to define a partial relation such that: given two sets of attributes A_1 and A_2, i.e. $U/IND(A_1) \preceq U/IND(A_2)$ (or $U/IND(A_2) \succeq U/IND(A_1)$) if and only if, for each $Y_i \in U/IND(A_1)$, there exists $X_j \in U/IND(A_2)$ such that $Y_i \subseteq X_j$ where $U/IND(A_1) = \{Y_1, Y_2, \cdots\}$ and $U/IND(A_2) = \{X_1, X_2, \cdots\}$ are partitions induced by indiscernibility relations $IND(A_1)$ and $IND(A_2)$ respectively. In this case, we say that A_2 is coarser than A_1, or A_1 is finer than A_2. If $U/IND(A_1) \preceq U/IND(A_2)$ and $U/IND(A_1) \neq U/IND(A_2)$, we say A_2 is strictly coarser than A_1 (or A_1 is strictly finer than A_2), denoted by $U/IND(A_1) \prec U/IND(A_2)$ (or $U/IND(A_2) \succ U/IND(A_1)$).

Theorem 2. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, suppose that $U/IND(A_1) \preceq U/IND(A_2) \preceq \cdots \preceq U/IND(A_m)$, then we have*

$$\underline{\sum_{i=1}^{m} A_i}^O(X) = \underline{A_1}(X); \tag{7}$$

$$\overline{\sum_{i=1}^{m} A_i}^O(X) = \overline{A_1}(X). \tag{8}$$

The above theorem tells us that if there is a partial relation among the partitions, then the optimistic multigranulation rough set is equivalent to the rough set in terms of the *finest* indiscernibility relation.

Theorem 3. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT, \forall X \subseteq U$, then we have following properties about the optimistic multigranulation rough approximations:*

1. $\underline{\sum_{i=1}^m A_i}^O(X) \subseteq X \subseteq \overline{\sum_{i=1}^m A_i}^O(X)$;
2. $\underline{\sum_{i=1}^m A_i}^O(\emptyset) = \overline{\sum_{i=1}^m A_i}^O(\emptyset) = \emptyset, \underline{\sum_{i=1}^m A_i}^O(U) = \overline{\sum_{i=1}^m A_i}^O(U) = U$;
3. $X \subseteq Y \Rightarrow \underline{\sum_{i=1}^m A_i}^O(X) \subseteq \underline{\sum_{i=1}^m A_i}^O(Y), \overline{\sum_{i=1}^m A_i}^O(X) \subseteq \overline{\sum_{i=1}^m A_i}^O(Y)$;
4. $\underline{\sum_{i=1}^m A_i}^O(X) = \bigcup_{i=1}^m \underline{A_i}(X), \overline{\sum_{i=1}^m A_i}^O(X) = \bigcap_{i=1}^m \overline{A_i}(X)$;

5. $\underline{\sum_{i=1}^{m} A_i}^{O}(\sim X) = \sim \left(\overline{\sum_{i=1}^{m} A_i}^{O}(X)\right), \overline{\sum_{i=1}^{m} A_i}^{O}(\sim X) = \sim \left(\underline{\sum_{i=1}^{m} A_i}^{O}(X)\right)$.

Theorem 4. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, then $\forall X_1, X_2, \cdots, X_j \subseteq U$, the optimistic multigranulation rough set has the following properties:*

1. $\underline{\sum_{i=1}^{m} A_i}^{O}\left(\bigcap_{j=1}^{n} X_j\right) = \bigcup_{i=1}^{m}\left(\bigcap_{j=1}^{n} \underline{A_i}(X_j)\right), \overline{\sum_{i=1}^{m} A_i}^{O}\left(\bigcup_{j=1}^{n} X_j\right) = \bigcap_{i=1}^{m}\left(\bigcup_{j=1}^{n} \overline{A_i}(X_j)\right)$;
2. $\underline{\sum_{i=1}^{m} A_i}^{O}\left(\bigcap_{j=1}^{n} X_j\right) \subseteq \bigcap_{j=1}^{n}\left(\underline{\sum_{i=1}^{m} A_i}^{O}(X_j)\right), \overline{\sum_{i=1}^{m} A_i}^{O}\left(\bigcup_{j=1}^{n} X_j\right) \supseteq \bigcup_{j=1}^{n}\left(\overline{\sum_{i=1}^{m} A_i}^{O}(X_j)\right)$;
3. $\underline{\sum_{i=1}^{m} A_i}^{O}\left(\bigcup_{j=1}^{n} X_j\right) \supseteq \bigcup_{j=1}^{n}\left(\underline{\sum_{i=1}^{m} A_i}^{O}(X_j)\right), \overline{\sum_{i=1}^{m} A_i}^{O}\left(\bigcap_{j=1}^{n} X_j\right) \subseteq \bigcap_{j=1}^{n}\left(\overline{\sum_{i=1}^{m} A_i}^{O}(X_j)\right)$.

Theorem 5. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, suppose that $A = A_1 \cup A_2 \cup \cdots \cup A_m$, then $\forall X \subseteq U$, we have*

$$\underline{\sum_{i=1}^{m} A_i}^{O}(X) \subseteq \underline{A}(X), \overline{\sum_{i=1}^{m} A_i}^{O}(X) \supseteq \overline{A}(X).$$

Theorem 5 shows that the optimistic multigranulation lower approximation is smaller than Pawlak's lower approximation while the optimistic multigranulation upper approximation is greater than Pawlak's upper approximation.

3. Pessimistic multigranulation rough sets

In decision making analysis, "Seeking common ground while eliminating differences" (SCED) is one of usual decision strategy. This strategy argues that one reserves common decisions while deleting inconsistent decisions, which can be seen as a conservative decision strategy. Based on this consideration, in this section, we propose a new multigranulation rough set, called pessimistic rough decision, and investigate its properties.

3.1. Pessimistic multigranulation rough set model and properties

Based on the SCED strategy, the following definition gives the formal representation of lower/upper approximation in the context of multi granular structures.

Definition 2. Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, then $\forall X \subseteq U$, the pessimistic multigranulation lower and upper approximations are denoted by $\underline{\sum_{i=1}^{m} A_i}^{P}(X)$ and $\overline{\sum_{i=1}^{m} A_i}^{P}(X)$, respectively,

$$\underline{\sum_{i=1}^{m} A_i}^{P}(X) = \{x \in U : \ [x]_{A_1} \subseteq X \wedge [x]_{A_2} \subseteq X \wedge \cdots \wedge [x]_{A_m} \subseteq X\}; \tag{9}$$

$$\overline{\sum_{i=1}^{m} A_i}^{P}(X) = \sim \left(\underline{\sum_{i=1}^{m} A_i}^{P}(\sim X)\right). \tag{10}$$

By the lower approximation $\underline{\sum_{i=1}^{m} A_i}^{P}(X)$ and upper approximation $\overline{\sum_{i=1}^{m} A_i}^{P}(X)$, the pessimistic multigranulation boundary region of X is

$$BN^{P}_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^{P}(X) - \underline{\sum_{i=1}^{m} A_i}^{P}(X). \tag{11}$$

Theorem 6. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, then $\forall X \subseteq U$, we have*

$$\overline{\sum_{i=1}^{m} A_i}^{P}(X) = \{x \in U : \ [x]_{A_1} \cap X \neq \emptyset \vee [x]_{A_2} \cap X \neq \emptyset \vee \cdots \vee [x]_{A_m} \cap X \neq \emptyset\}. \tag{12}$$

Proof. By Definition 2, we have

$$
\begin{aligned}
x \in \overline{\sum_{i=1}^{m} A_i{}^P}(X) \iff \quad & x \notin \underline{\sum_{i=1}^{m} A_i{}^P}(\sim X) \\
\iff \quad & [x]_{A_1} \nsubseteq (\sim X) \vee [x]_{A_2} \nsubseteq (\sim X) \vee \cdots \vee [x]_{A_m} \nsubseteq (\sim X) \\
\iff \quad & [x]_{A_1} \cap X \neq \emptyset \vee [x]_{A_2} \cap X \neq \emptyset \vee \cdots \vee [x]_{A_m} \cap X \neq \emptyset. \quad \square
\end{aligned}
$$

Different from the upper approximation of optimistic multigranulation rough set, the upper approximation of pessimistic multigranulation rough set is represented as a set in which objects have non-empty intersection with the target in terms of at least one granular structure.

Theorem 7. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, suppose that $U/IND(R_1) \preceq U/IND(R_2) \preceq \cdots \preceq U/IND(R_m)$, then we have*

$$\underline{\sum_{i=1}^{m} A_i{}^P}(X) = \underline{A_m}(X); \tag{13}$$

$$\overline{\sum_{i=1}^{m} A_i{}^P}(X) = \overline{A_m}(X). \tag{14}$$

Proof. $\forall x \in \underline{\sum_{i=1}^{m} A_i{}^P}(X)$, then we have $[x]_{A_i} \subseteq X$ for each $i = 1, 2, \cdots, m$, it follows that $x \in \underline{A_m}(X)$ because $[x]_{A_m} \subseteq X$.

$\forall x \in \underline{A_m}(X)$, we have $[x]_{A_m} \subseteq X$. Moreover, since $U/IND(R_1) \preceq U/IND(R_2) \preceq \cdots \preceq U/IND(R_m)$, then we have $[x]_{A_1} \subseteq [x]_{A_2} \subseteq [x]_{A_m}$, from which we can conclude that $[x]_{A_1} \subseteq [x]_{A_2} \subseteq [x]_{A_m} \subseteq X$. Then by the definition of pessimistic multigranulation lower approximation, we have $x \in \underline{\sum_{i=1}^{m} A_i{}^P}(X)$.

From discussions above, we can conclude that $\underline{\sum_{i=1}^{m} A_i{}^P}(X) = \underline{A_m}(X)$. Similarly, it is not difficult to prove that $\overline{\sum_{i=1}^{m} A_i{}^P}(X) = \overline{A_m}(X)$. $\square$

The above theorem tells us that if there is a partial relation among the partitions, then the pessimistic multigranulation rough set is equivalent to the rough set in terms of the *coarsest* equivalence relation.

Theorem 8. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT, \forall X, Y \subseteq U$, then we have following properties about the pessimistic multigranulation rough approximation:*

1. $\underline{\sum_{i=1}^{m} A_i{}^P}(X) \subseteq X \subseteq \overline{\sum_{i=1}^{m} A_i{}^P}(X)$;
2. $\underline{\sum_{i=1}^{m} A_i{}^P}(\emptyset) = \overline{\sum_{i=1}^{m} A_i{}^P}(\emptyset) = \emptyset, \underline{\sum_{i=1}^{m} A_i{}^P}(U) = \overline{\sum_{i=1}^{m} A_i{}^P}(U) = U$;
3. $X \subseteq Y \Rightarrow \underline{\sum_{i=1}^{m} A_i{}^P}(X) \subseteq \underline{\sum_{i=1}^{m} A_i{}^P}(Y), \overline{\sum_{i=1}^{m} A_i{}^P}(X) \subseteq \overline{\sum_{i=1}^{m} A_i{}^P}(Y)$;
4. $\underline{\sum_{i=1}^{m} A_i{}^P}(X) = \bigcap_{i=1}^{m} \underline{A_i}(X), \overline{\sum_{i=1}^{m} A_i{}^P}(X) = \bigcup_{i=1}^{m} \overline{A_i}(X)$;
5. $\underline{\sum_{i=1}^{m} A_i{}^P}(\sim X) = \sim \left(\overline{\sum_{i=1}^{m} A_i{}^P}(X)\right), \overline{\sum_{i=1}^{m} A_i{}^P}(\sim X) = \sim \left(\underline{\sum_{i=1}^{m} A_i{}^P}(X)\right)$.

Proof.

1. $\forall x \in \underline{\sum_{i=1}^{m} A_i{}^P}(X)$, then by the definition of pessimistic multigranulation lower approximation, we have $[x]_{A_i} \subseteq X$ for each $i = 1, 2, \cdots, m$. Since indiscernibility relation is reflexive, then we have $x \in [x]_{A_i}$ for each $i = 1, 2, \cdots, m$, it follows that $x \in X$, i.e. $\underline{\sum_{i=1}^{m} A_i{}^P}(X) \subseteq X$.$\forall x \in X$, since indiscernibility relation is reflexive, then we have $x \in [x]_{A_i}$ for each $i = 1, 2, \cdots, m$, it follows that $[x]_{A_i} \cap X \neq \emptyset$ for each $i = 1, 2, \cdots, m$. By Theorem 6 we have $x \in \overline{\sum_{i=1}^{m} A_i{}^P}(X)$, i.e. $X \subseteq \overline{\sum_{i=1}^{m} A_i{}^P}(X)$.
2. $\emptyset \subseteq \underline{\sum_{i=1}^{m} A_i{}^P}(\emptyset)$ holds obviously since the empty set is included into each set. Moreover, by the proof of 1 we know that $\underline{\sum_{i=1}^{m} A_i{}^P}(\emptyset) \subseteq \emptyset$. Therefore, $\underline{\sum_{i=1}^{m} A_i{}^P}(\emptyset) = \emptyset$ holds.$\emptyset \subseteq \overline{\sum_{i=1}^{m} A_i{}^P}(\emptyset)$ holds obviously since the empty set is included into each set. $\forall x \notin \emptyset$, we have $x \in U$. Since the indiscernibility relation is reflexive, then we have $x \in [x]_{A_i}$ for each $i = 1, 2, \cdots, m$, thus, $[x]_{A_i} \cap \emptyset = \emptyset$ for each $i = 1, 2, \cdots, m$. By Theorem 6, we know that $x \notin \overline{\sum_{i=1}^{m} A_i{}^P}(\emptyset)$, it follows that $\overline{\sum_{i=1}^{m} A_i{}^P}(\emptyset) \subseteq \emptyset$. From discussions above, we have $\overline{\sum_{i=1}^{m} A_i{}^P}(\emptyset) = \emptyset$.
   Similarly, it is not difficult to prove $\underline{\sum_{i=1}^{m} A_i{}^P}(U) = \overline{\sum_{i=1}^{m} A_i{}^P}(U) = U$.

3. $\forall x \in \underline{\sum_{i=1}^{m} A_i}^P(X)$, then by the definition of pessimistic multigranulation lower approximation, we have $[x]_{A_i} \subseteq X$ for each $i = 1, 2, \cdots, m$. Since $X \subseteq Y$, then $[x]_{A_i} \subseteq Y$ for each $i = 1, 2, \cdots, m$, it follows that $x \in \underline{\sum_{i=1}^{m} A_i}^P(Y)$, i.e. $\underline{\sum_{i=1}^{m} A_i}^P(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^P(Y)$. Similarly, it is not difficult to prove that $\overline{\sum_{i=1}^{m} A_i}^P(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^P(Y)$.
4. $\forall x \in \underline{\sum_{i=1}^{m} A_i}^P(X)$, then by the definition of pessimistic multigranulation lower approximation, we have $[x]_{A_i} \subseteq X$ for each $i = 1, 2, \cdots, m$. Moreover, by the definition of classical lower approximation, we have $x \in \underline{A_i}(X)$ for each $i = 1, 2, \cdots, m$, it follows that $x \in \bigcap_{i=1}^{m} \underline{A_i}(X)$, i.e. $\underline{\sum_{i=1}^{m} A_i}^P(X) \subseteq \bigcap_{i=1}^{m} \underline{A_i}(X)$.$\forall x \in \bigcap_{i=1}^{m} \underline{A_i}(X)$, we have $[x]_{A_i} \subseteq X$ for each $i = 1, 2, \cdots, m$. Then by the definition of pessimistic multigranulation lower approximation, $x \in \underline{\sum_{i=1}^{m} A_i}^P(X)$ holds obviously, i.e. $\bigcap_{i=1}^{m} \underline{A_i}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^P(X)$.
   From discussions above, we have $\underline{\sum_{i=1}^{m} A_i}^P(X) = \bigcap_{i=1}^{m} \underline{A_i}(X)$. Similarly, it is not difficult to prove $\overline{\sum_{i=1}^{m} A_i}^P(X) = \bigcup_{i=1}^{m} \overline{A_i}(X)$.
5. $\forall x \in \underline{\sum_{i=1}^{m} A_i}^P(\sim X)$, then by the definition of pessimistic multigranulation lower approximation, we have $[x]_{A_i} \subseteq (\sim X)$ for each $i = 1, 2, \cdots, m$, it follows that $[x]_{A_i} \cap X = \emptyset$ for each $i = 1, 2, \cdots, m$. By Theorem 6, we can conclude that $x \notin \overline{\sum_{i=1}^{m} A_i}^P(X)$, i.e. $\underline{\sum_{i=1}^{m} A_i}^P(\sim X) \subseteq \sim \left(\overline{\sum_{i=1}^{m} A_i}^P(X)\right)$.$\forall x \notin \overline{\sum_{i=1}^{m} A_i}^P(X)$, by Theorem 6, we have $[x]_{A_i} \cap X = \emptyset$ for each $i = 1, 2, \cdots, m$, i.e. $[x]_{A_i} \subseteq (\sim X)$ for each $i = 1, 2, \cdots, m$, then by the definition of pessimistic multigranulation lower approximation, we can conclude that $x \in \underline{\sum_{i=1}^{m} A_i}^P(\sim X)$, i.e. $\sim \left(\overline{\sum_{i=1}^{m} A_i}^P(X)\right) \subseteq \underline{\sum_{i=1}^{m} A_i}^P(\sim X)$.
   From discussions above, we have $\underline{\sum_{i=1}^{m} A_i}^P(\sim X) = \sim \left(\overline{\sum_{i=1}^{m} A_i}^P(X)\right)$. Similarly, it is not difficult to prove that $\overline{\sum_{i=1}^{m} A_i}^P(\sim X) = \sim \left(\underline{\sum_{i=1}^{m} A_i}^P(X)\right)$. □

Theorem 9. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, then $\forall X_1, X_2, \cdots, X_j \subseteq U$, the pessimistic multigranulation rough set has the following properties:*

1. $\underline{\sum_{i=1}^{m} A_i}^P\left(\bigcap_{j=1}^{n} X_j\right) = \bigcap_{i=1}^{m}\left(\bigcap_{j=1}^{n} \underline{A_i}(X_j)\right)$, $\overline{\sum_{i=1}^{m} A_i}^P\left(\bigcup_{j=1}^{n} X_j\right) = \bigcup_{i=1}^{m}\left(\bigcup_{j=1}^{n} \overline{A_i}(X_j)\right)$;
2. $\underline{\sum_{i=1}^{m} A_i}^P\left(\bigcap_{j=1}^{n} X_j\right) = \bigcap_{j=1}^{n}\left(\underline{\sum_{i=1}^{m} A_i}^P(X_j)\right)$, $\overline{\sum_{i=1}^{m} A_i}^P\left(\bigcup_{j=1}^{n} X_j\right) = \bigcup_{j=1}^{n}\left(\overline{\sum_{i=1}^{m} A_i}^P(X_j)\right)$;
3. $\underline{\sum_{i=1}^{m} A_i}^P\left(\bigcup_{j=1}^{n} X_j\right) \supseteq \bigcup_{j=1}^{n}\left(\underline{\sum_{i=1}^{m} A_i}^P(X_j)\right)$, $\overline{\sum_{i=1}^{m} A_i}^P\left(\bigcap_{j=1}^{n} X_j\right) \subseteq \bigcap_{j=1}^{n}\left(\overline{\sum_{i=1}^{m} A_i}^P(X_j)\right)$.

Proof.

1. $\forall x \in \underline{\sum_{i=1}^{m} A_i}^P\left(\bigcap_{j=1}^{n} X_j\right)$, then by the definition of pessimistic multigranulation lower approximation, we have $[x]_{A_i} \subseteq \bigcap_{j=1}^{n} X_j$ for each $i = 1, 2, \cdots, m$, it follows that $[x]_{A_i} \subseteq X_j$ for each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, from which we can conclude that $x \in \bigcap_{j=1}^{n} \underline{A_i}(X_j)$ for each $i = 1, 2, \cdots, m$, i.e. $x \in \bigcap_{i=1}^{m}\left(\bigcap_{j=1}^{n} \underline{A_i}(X_j)\right)$.$\forall x \in \bigcap_{i=1}^{m}\left(\bigcap_{j=1}^{n} \underline{A_i}(X_j)\right)$, we have $x \in \bigcap_{j=1}^{n} \underline{A_i}(X_j)$ for each $i = 1, 2, \cdots, m$. Moreover, since $x \in \bigcap_{j=1}^{n} \underline{A_i}(X_j)$, then $x \in \underline{A_i}(X_j)$ for each $j = 1, 2, \cdots, n$, from which we can conclude that $[x]_{A_i} \subseteq X_j$ for each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. Thus, $[x]_{A_i} \subseteq \bigcap_{j=1}^{n} X_j$ for each $i = 1, 2, \cdots, m$, by the definition of pessimistic multigranulation lower approximation, we have $x \in \underline{\sum_{i=1}^{m} A_i}^P\left(\bigcap_{j=1}^{n} X_j\right)$.
   From discussions above, we have $\underline{\sum_{i=1}^{m} A_i}^P\left(\bigcap_{j=1}^{n} X_j\right) = \bigcap_{i=1}^{m}\left(\bigcap_{j=1}^{n} \underline{A_i}(X_j)\right)$. Similarly, it is not difficult to prove $\overline{\sum_{i=1}^{m} A_i}^P\left(\bigcup_{j=1}^{n} X_j\right) = \bigcup_{i=1}^{m}\left(\bigcup_{j=1}^{n} \overline{A_i}(X_j)\right)$.
2. By the proof of 4 in Theorem 8, we have $\underline{\sum_{i=1}^{m} A_i}^P\left(\bigcap_{j=1}^{n} X_j\right) = \bigcap_{i=1}^{m} \underline{A_i}\left(\bigcap_{j=1}^{n} X_j\right)$. By the basic property of Pawlak's rough set, we know $\bigcap_{i=1}^{m} \underline{A_i}\left(\bigcap_{j=1}^{n} X_j\right) = \bigcap_{j=1}^{n} \bigcap_{i=1}^{m} \underline{A_i}(X_j)$. Since $\bigcap_{i=1}^{m} \underline{A_i}(X_j) = \underline{\sum_{i=1}^{m} A_i}^P(X_j)$ holds through the proof of 4 in Theorem 8, then $\underline{\sum_{i=1}^{m} A_i}^P\left(\bigcap_{j=1}^{n} X_j\right) = \bigcap_{i=1}^{m} \underline{A_i}\left(\bigcap_{j=1}^{n} X_j\right) = \bigcap_{j=1}^{n} \bigcap_{i=1}^{m} \underline{A_i}(X_j) = \bigcap_{j=1}^{n}\left(\underline{\sum_{i=1}^{m} A_i}^P(X_j)\right)$.
   Similarly, it is not difficult to prove $\overline{\sum_{i=1}^{m} A_i}^P\left(\bigcup_{j=1}^{n} X_j\right) = \bigcup_{j=1}^{n}\left(\overline{\sum_{i=1}^{m} A_i}^P(X_j)\right)$.
3. $\forall x \in \bigcup_{j=1}^{n}\left(\underline{\sum_{i=1}^{m} A_i}^P(X_j)\right)$, then $\exists X_k (k \in \{1, 2, \cdots, n\})$ such that $x \in \underline{\sum_{i=1}^{m} A_i}^P(X_k)$. By the definition of pessimistic multigranulation lower approximation, $[x]_{A_i} \subseteq X_k$ for each $i = 1, 2, \cdots, m$, i.e. $[x]_{A_i} \subseteq \bigcup_{j=1}^{n} X_j$ for each $i = 1, 2, \cdots, m, x \in \underline{\sum_{i=1}^{m} A_i}^P\left(\bigcup_{j=1}^{n} X_j\right)$, from which we can conclude that $\underline{\sum_{i=1}^{m} A_i}^P\left(\bigcup_{j=1}^{n} X_j\right) \supseteq \bigcup_{j=1}^{n}\left(\underline{\sum_{i=1}^{m} A_i}^P(X_j)\right)$. Similarly, it is not difficult to prove $\overline{\sum_{i=1}^{m} A_i}^P\left(\bigcap_{j=1}^{n} X_j\right) \subseteq \bigcap_{j=1}^{n}\left(\overline{\sum_{i=1}^{m} A_i}^P(X_j)\right)$. □

Theorem 10. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, suppose that $A = A_1 \cup A_2 \cup \cdots \cup A_m$, then $\forall X \subseteq U$, we have*

$$\underline{\sum_{i=1}^{m} A_i}^P(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^O(X) \subseteq \underline{A}(X), \overline{\sum_{i=1}^{m} A_i}^P(X) \supseteq \overline{\sum_{i=1}^{m} A_i}^O(X) \supseteq \overline{A}(X).$$

Proof. Following Theorem 5, it must be proved that $\underline{\sum_{i=1}^{m} A_i}^P(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^O(X)$ and $\overline{\sum_{i=1}^{m} A_i}^P(X) \supseteq \overline{\sum_{i=1}^{m} A_i}^O(X)$.$\forall x \in \underline{\sum_{i=1}^{m} A_i}^P(X)$, by Definition 2, we have $[x]_{A_i} \subseteq X$ $(\forall i \in \{1, 2, \cdots, m\})$, then by Definition 1, $x \in \underline{\sum_{i=1}^{m} A_i}^O(X)$ holds.$\forall x \in \overline{\sum_{i=1}^{m} A_i}^O(X)$, by Theorem 1, we have $[x]_{A_i} \cap X \neq \emptyset$ $(\forall i \in \{1, 2, \cdots, m\})$, then by Theorem 3, $x \in \overline{\sum_{i=1}^{m} A_i}^P(X)$ holds. □

The above theorem tells us that the pessimistic multigranulation lower approximation is smaller than Pawlak's lower approximation while the pessimistic multigranulation upper approximation is greater than Pawlak's upper approximation. Moreover, the pessimistic multigranulation lower approximation is smaller than optimistic multigranulation lower approximation while the pessimistic multigranulation upper approximation is greater than optimistic multigranulation upper approximation.

Remark. From the above theorem, it can be seen that the "distance" between the pessimistic multigranulation lower approximation and its upper approximation is largest, which leads to larger boundary region (bigger uncertainty) for a given subset X than optimistic multigranulation rough set and Pawlak's rough set. However, it is not disappointed, which is because that the uncertainty is determined by each given multigranulation fusion strategy. The pessimistic multigranulation rough set adopts the decision strategy "Seeking common ground while eliminating differences", and the decision induced by it must be consistent with every decision maker's viewpoint. In fact, the pessimistic decision and optimistic decision can be seen as two extreme cases of information fusion based on multigranulation rough sets.

3.2. Rough memberships

In the rough set literature, rough membership function introduced in [20] can be used to measure degrees of inclusion of equivalence classes into subsets of the universe.

Definition 3. Let I be an information system in which $A \subseteq AT, \forall X \subseteq U$, the rough membership of x in X is denoted by $\mu_X^A(x)$ such that

$$\mu_X^A(x) = \frac{|\, [x]_A \cap X \,|}{|\, [x]_A \,|}. \tag{15}$$

In Pawlak's rough set model, there is a direct relationship between the rough approximation and the membership such that

$$\mu_X^A(x) = 1 \Longleftrightarrow x \in \underline{A}(X); \tag{16}$$

$$0 < \mu_X^A(x) \leq 1 \Longleftrightarrow x \in \overline{A}(X). \tag{17}$$

It should be noticed that since more than one indiscernibility relations are used in multigranulation rough set approach, the re-definition of the rough membership has become a necessity.

Definition 4. Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT, \forall X \subseteq U$, the maximal and minimal rough memberships of x in X are denoted by $\eta_X^A(x)$ and $\theta_X^A(x)$, respectively, where

$$\eta_X^{\sum_{i=1}^{m} A_i}(x) = max_{i=1}^{m} \mu_X^{A_i}(x); \tag{18}$$

$$\theta_X^{\sum_{i=1}^{m} A_i}(x) = min_{i=1}^{m} \mu_X^{A_i}(x). \tag{19}$$

Theorem 11. *Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT, \forall X \subseteq U$, we have*

1. $\eta_X^{\sum_{i=1}^{m} A_i}(x) = 1 \Longleftrightarrow x \in \underline{\sum_{i=1}^{m} A_i^O}(X)$;
2. $0 < \theta_X^{\sum_{i=1}^{m} A_i}(x) \leq 1 \Longleftrightarrow x \in \overline{\sum_{i=1}^{m} A_i^O}(X)$;
3. $\theta_X^{\sum_{i=1}^{m} A_i}(x) = 1 \Longleftrightarrow x \in \underline{\sum_{i=1}^{m} A_i^P}(X)$;
4. $0 < \eta_X^{\sum_{i=1}^{m} A_i}(x) \leq 1 \Longleftrightarrow x \in \overline{\sum_{i=1}^{m} A_i^P}(X)$.

Proof. For the term 1, we have that

$$\begin{aligned}\eta_X^{\sum_{i=1}^m}(x) = 1 \iff & max_{i=1}^m \mu_X^{A_i}(x) = 1\\ \iff & \exists i \in \{1,2,\cdots,m\} \text{s.t.} [x]_{A_i} \subseteq X\\ \iff & x \in \underline{\sum_{i=1}^m A_i}^O(X)\end{aligned}$$

Similarly, the terms 2, 3, and 4 also can be proved. □

4. Rules in multigranulation rough sets

The end result of the rough set model is a representation of the information contained in the data system considered in terms of "if···then···" decision rules. The decision rules can be generated from the decision system in the rough set approach. A decision system is an information system such that $I = (U, AT \cup D)$ in which AT is the set of condition attributes, while D is the set of decision attributes. In our paper, to simplify our discussion, we only consider one decision attribute d and then the decision system can be represented by $I = (U, AT \cup \{d\})$. Generally speaking, we may assume that such decision attribute determines a partition on the universe of discourse, i.e $U/IND(\{d\}) = \{X_1, X_2, \cdots, X_l\}$.

Following the optimistic and pessimistic multigranulation rough set approaches mentioned above, suppose that $A = \{a_1, a_2, \cdots, a_m\}$, then $\forall x \in U$, we may desire to generate the following decision rules from a decision system:

- "OR" decision rule: $r_x^\vee:\ a_1(y) = a_1(x) \vee a_2(y) = a_2(x) \vee \cdots \vee a_m(y) = a_m(x) \to d(y) = d(x)$.

Obviously, the "OR" decision rule $r_x^\vee$ is different from the rules which can be induced from Pawlak's rough set model because the condition part of $r_x^\vee$ is composed by the logical connective "$\vee$" (disjunction). In essence, the restriction of such decision rule is weaker than that of decision rules in Pawlak's rough set theory, since intersection operations among equivalence classes need not to be performed in optimistic multigranulation rough set.

Generally speaking, the "OR" decision rule $r_x^\vee$ can be decomposed into a family of decision rules such that

$$\begin{aligned}& r_x^1:\ a_1(y) = a_1(x) \to d(y) = d(x)\\ & r_x^2:\ a_2(y) = a_2(x) \to d(y) = d(x)\\ & \vdots\\ & r_x^m:\ a_m(y) = a_m(x) \to d(y) = d(x)\end{aligned}$$

The certainty factor of the decision rule $r_x^\vee$ is defined as

$$C(r_x^\vee) = \max_{i=1}^m (C(r_x^i)) \tag{20}$$

where $C(r_x^i)$ $(1 \leqslant i \leqslant m)$ is the certainty factor of the decomposed decision rule r_x^i, that is,

$$C(r_x^i) = \frac{|\ [x]_{a_i} \cap [x]_d\ |}{|\ [x]_{a_i}\ |}, \tag{21}$$

in which $[x]_{a_i}$ is the equivalence class of x in terms of the condition attribute a_i, $[x]_d$ is the equivalence class of x in terms of the decision attribute d, $|\ X\ |$ is the cardinal number of the set X.

Similar to the decision rules in Pawlak's rough set theory, in multigranulation rough set theory, the rules $r_x^\vee$ are referred to as certain if and only if $C(r_x^\vee) = 1$; the rules $r_x^\vee$ are referred to as possible if and only if $0 < C(r_x^\vee) < 1$.

Theorem 12. *Let I be a decision system in which $A = \{a_1, a_2, \cdots, a_m\} \subseteq AT$, then $\forall x \in U$*

1. $x \in \underline{\sum_{i=1}^m a_i}^O([x]_d) \iff C(r_x^\vee) = 1$;
2. $x \in BN^P_{\sum_{i=1}^m a_i}([x]_d) \iff 0 < C(r_x^\vee) < 1$;

Proof. We only prove 1, the proofs of 2 is similar to the proofs of 1.$\forall x \in U$,

$$\begin{aligned}x \in \underline{\sum_{i=1}^m a_i}^O([x]_d) \iff & \exists a_i \in A \text{ s.t.} [x]_{a_i} \subseteq [x]_d\\ \iff & \exists a_i \in A \text{ s.t.} C(r_x^i) = \frac{|\ [x]_{a_i} \cap [x]_d\ |}{|\ [x]_{a_i}\ |} = 1\\ \iff & C(r_x^\vee) = \max_{i=1}^m (C(r_x^i)) = 1 \quad \square\end{aligned}$$

By the above theorem, we can draw the following conclusions:

1. The certain "OR" rules are supported by the objects in optimistic multigranulation lower approximation;
2. The possible "OR" rules are supported by the objects in pessimistic multigranulation boundary region.

5. Attribute reduction

Intuitively, some attributes are not significant in a representation and their removal has no real impact on the value of the representation of elements. If it is not significant, one can simply remove an attribute for further consideration.

Definition 5. Let I be a decision system in which $A = \{a_1, a_2, \cdots, a_m\} \subseteq AT = \{a_1, a_2, \cdots, a_n\}$. We define

$$\underline{AT}^P(d) = \left\{ \underline{\sum_{i=1}^{n} a_i{}^P}(X_1), \underline{\sum_{i=1}^{n} a_i{}^P}(X_2), \cdots, \underline{\sum_{i=1}^{n} a_i{}^P}(X_l) \right\}; \tag{22}$$

$$\overline{AT}^P(d) = \left\{ \overline{\sum_{i=1}^{n} a_i{}^P}(X_1), \overline{\sum_{i=1}^{n} a_i{}^P}(X_2), \cdots, \overline{\sum_{i=1}^{n} a_i{}^P}(X_l) \right\}; \tag{23}$$

$$BN_{AT}^P(d) = \left\{ BN^P_{\sum_{i=1}^{n} A_i}(X_1), BN^P_{\sum_{i=1}^{n} A_i}(X_2), \cdots, BN^P_{\sum_{i=1}^{n} A_i}(X_l) \right\}; \tag{24}$$

1. A is referred to as the pessimistic multigranulation lower approximate distribution reduct if and only if $\underline{A}^P(d) = \underline{AT}^P(d)$ and $\underline{B}^P(d) \neq \underline{AT}^P(d)$ for each $B \subset A$;
2. A is referred to as the pessimistic multigranulation upper approximate distribution reduct if and only if $\overline{A}^P(d) = \overline{AT}^P(d)$ and $\overline{B}^P(d) \neq \overline{AT}^P(d)$ for each $B \subset A$;
3. A is referred to as the pessimistic multigranulation boundary region distribution reduct if and only if $BN_A^P(d) = BN_{AT}^P(d)$ and $BN_B^P(d) \neq BN_{AT}^P(d)$ for each $B \subset A$.

Since the pessimistic multigranulation rough set model mainly considers the lower approximation and the upper approximation of a target concept by multiple equivalence relations, in the following, we introduce a measure of importance of condition attributes with respect to decision attributes in a decision system.

Let I be a decision system in which $A = \{a_1, a_2, \cdots, a_m\} \subseteq AT = \{a_1, a_2, \cdots, a_n\}$. A measure of importance of condition attributes $A \subseteq AT$ with respect to decision attributes d in pessimistic MGRS in terms of the under approximation and the upper approximation can be divided into two forms: an *importance measure of the lower approximation* and an *importance measure of the upper approximation*.

Let I be a decision system in which $A = \{a_1, a_2, \cdots, a_m\} \subseteq AT = \{a_1, a_2, \cdots, a_n\}$. Given a condition attribute $a \in A$ and $X \in U/\{d\}$. Firstly, we give two preliminary definitions in the following.

Definition 6. We say that a is *lower approximation significant* in A with respect to X if $\underline{\sum_{i=1}^{m} a_i{}^P}X \subset \underline{\sum_{i=1,a_i \neq a}^{m} a_i{}^P}X$ $(a_i \in A)$, and that a is not lower approximation significant in A with respect to X if $\underline{\sum_{i=1}^{m} a_i{}^P}X = \underline{\sum_{i=1,a_i \neq a}^{m} a_i{}^P}X$ $(a_i \in A)$.

Definition 7. We say that a is *upper approximation significant* in A with respect to X if $\overline{\sum_{i=1}^{m} a_i{}^P}X \supset \overline{\sum_{i=1,a_i \neq a}^{m} a_i{}^P}X$ $(a_i \in A)$, and that a is not upper approximation significant in A with respect to X if $\overline{\sum_{i=1}^{m} a_i{}^P}X = \overline{\sum_{i=1,a_i \neq a}^{m} a_i{}^P}X$ $(a_i \in A)$.

We introduce a quantitative measure for the significance as follows.

The *importance measure of the lower approximation* of condition attributes $A \subseteq AT$ with respect to decision attributes D in MGRS is defined as

$$S_A(d) = \frac{\sum \left\{ \left| \underline{\sum_{i=1,a_i \neq a}^{m} a_i{}^P}X \setminus \underline{\sum_{i=1}^{m} a_i{}^P}X \right| : \ X \in U/\{d\} \right\}}{|\ U\ |}, \tag{25}$$

where the attributes $A = \{a_1, a_2, \cdots, a_m\}$, and d is the decision attribute.

The *importance measure of the upper approximation* of condition attributes $A \subseteq AT$ with respect to decision attributes D in MGRS is defined as

$$S^A(d) = \frac{\sum \left\{ |\ \overline{\sum_{i=1}^{m} a_i{}^P}X \setminus \overline{\sum_{i=1,a_i \neq a}^{m} a_i{}^P}X\ |: \ X \in U/\{d\} \right\}}{|\ U\ |}, \tag{26}$$

where the attributes $A = \{a_1, a_2, \cdots, a_m\}$, and d is the decision attribute.

In particular, when $A=\{a\}, S_a(d)$ is the importance measure of the lower approximation of the attribute $a \in A$ with respect to d and $S^a(d)$ is the importance measure of the upper approximation of the attribute $a \in A$ with respect to d.

To compute the significance of an attribute a in A with respect to d, we need to compute m partitions $U/\{a_i\}$ $(i \leqslant m)$. The time complexity for computing each partition is $\mathbf{O}(|U|)$. So, the time complexity for computing m partitions is $\mathbf{O}(m|U|)$. Therefore, the time complexity of computing a lower approximation of $X \in U/\{d\}$ by A is $\mathbf{O}(m|U|^2)$.

From the above two definitions, we know the following:

- $S_A(D) \geqslant 0$ and $S^A(D) \geqslant 0$;
- attributes A with respect to d is the lower approximation significant if and only if $S_A(d)=0$; and
- attributes A with respect to d is the upper approximation significant if and only if $S^A(d)=0$.

Based on the above measures, we give the inner significance measure and the outer significance measure of an attribute for designing a heuristic attribute reduction algorithm. Simply, we only consider the pessimistic multigranulation lower approximate distribution reduct, which are defined as follows.

Definition 8. Let I be a decision system in which $A=\{a_1,a_2,\cdots,a_m\} \subseteq AT=\{a_1,a_2,\cdots,a_n\}$ and $\forall a \in A$. The significance measure of a in A is defined as

$$Sig^{inner}(a,A,d)=S_{A-\{a\}}(d)-S_A(d). \tag{27}$$

Definition 9. Let I be a decision system in which $A=\{a_1,a_2,\cdots,a_m\} \subseteq AT=\{a_1,a_2,\cdots,a_n\}$ and $\forall a \in AT-A$. The significance measure of a in A is defined as

$$Sig^{outer}(a,A,d)=S_A(d)-S_{A\cup\{a\}}(d). \tag{28}$$

Using each of these two significance measures of an attribute, we can design a forward greedy attribute reduction algorithm in the pessimistic multigranulation rough set model. It can be formally written as follows.

Algorithm 1. An attribute reduction algorithm in the pessimistic multigranulation rough set

Input: a decision table $I=(U, C\cup\{d\})$;
Output: One reduct *red*.
Step 1: $red \leftarrow \emptyset$; //*red* is the pool to conserve the selected attributes
Step 2: Compute $Sig^{inner}(a_k,C,d), k \leqslant |C|$;
Step 3: Put a_k into *red*, where $Sig^{inner}(a_k,C,d)>0$; // These attributes form the core of the given decision system
Step 4: While $S_{red}(d) \neq S_C(d)$ Do//This provides a stopping criterion
$\{red \leftarrow red \cup \{a_0\}$, where $Sig^{outer}(a_0,red,d)=max\{Sig^{outer}(a_k,red,d), a_k \in C-red\}\}$; $//Sig^{outer}(a_k,C,d)$ is the outer importance measure of the attribute a_k
Step 5: return *red* and end.

Computing the significance measure of an attribute is one of the key steps in the algorithm. The time complexity of computing the core in Step 2 is $O(|C||U|^2)$. In Step 5, we begin with the core and add an attribute with the maximal significance into the set in each stage until finding a reduct. This process is called a forward reduction algorithm whose time complexity is $O(\sum_{i=1}^{C}|U|^2(|C|-i+1)$. Thus the time complexity of Algorithm 1 is $O(|C||U|^2+\sum_{i=1}^{C}|U|^2(|C|-i+1)$.

The algorithm can obtain an attribute reduct from a given decision system, which provides the minimal attributes (or granular structures) that retain the lower approximation of a target decision unchanged in the context of pessimistic multigranulation rough sets.

Table 1a
The information system from site 1.

U	a_1	a_2	a_3
x_1	2	2	2
x_2	2	2	2
x_3	0	0	0
x_4	0	0	0
x_5	0	0	0
x_6	2	1	0
x_7	0	0	0
x_8	2	2	2
x_9	2	1	0
x_{10}	0	0	0

6. Case study

For an investor or decision maker, he may need to adopt a better one from some possible investment projects or find some directions from existing successful investment projects before investing. The purpose of this section is, through a venture investment issue, to illustrate the mechanism of the pessimistic rough set based decision and its applications.

In this section, we consider the site selection of a marketplace through using the proposed method in this paper. There are ten investment projects $x_i (i = 1, 2, \cdots, 10)$ can be considered, which evaluations are from online five sites. There are three factors to be considered, which are Locus (a_1), Investment (a_2) and Population density (a_3). Venture level is classified to three classes 0, 1 and 2. The bigger the value of venture level is, and the higher the venture of investment project is. Tables 1a, 1b, 1c, 1d, 1e are information systems about evaluating venture investment given by these five sites.

In these evaluation information systems, each one makes a decision independently, especially their attributes may be inconsistent. For this situation, the classical rough set model will be helpless. It is also very difficult that these information systems are combined into a entire information system. Therefore, In the following, we apply the pessimistic MGRS proposed by this paper for decision-making. Given a classification from real applications, $U/IND(D) = \{\{x_1, x_2, x_6, x_8, x_9\}, \{x_3, x_4, x_5, x_7, x_{10}\}\}$. Suppose that $Y_1 = \{x_1, x_2, x_6, x_8, x_9\}$ and $Y_2 = \{x_3, x_4, x_5, x_7, x_{10}\}$. For each site, we calculate its granular structure (partition) as follows.

For the information system from site 1 (wrt. S_1),

$$U/\{a_1, a_2, a_3\} = \{\{x_1, x_2, x_8\}, \{x_6, x_9\}, \{x_3, x_4, x_5, x_7, x_{10}\}\};$$

for the information system from site 2 (wrt. S_2),

$$U/\{a_1, a_2\} = \{\{x_1, x_2, x_6, x_8, x_9\}, \{x_4, x_7\}, \{x_3, x_5, x_{10}\}\};$$

for the information system from site 3 (wrt. S_3),

$$U/\{a_2, a_3\} = \{\{x_1, x_2, x_8, x_9\}, \{x_6, x_7\}, \{x_3, x_4, x_5, x_{10}\}\};$$

for the information system from site 4 (wrt. S_4),

$$U/\{a_1, a_3\} = \{\{x_1, x_2, x_6, x_8\}, \{x_4, x_9\}, \{x_3, x_5, x_7, x_{10}\}\};$$

and for the information system from site 5 (wrt. S_5),

$$U/\{a_1, a_2, a_3\} = \{\{x_1, x_2, x_8, x_9\}, \{x_4, x_6\}, \{x_3, x_5, x_7, x_{10}\}\}.$$

From Definition 2, one can obtain the following pessimistic multigranulation lower approximation of the classification, which is as follows

$\underline{\mathbf{S}}^P(D) = \{\{x_1, x_2, x_8\}, \{x_3, x_5, x_{10}\}\}$, where $\mathbf{S} = \{S_1, S_2, S3, S_4, S_5\}$.

In what follows, we compute the inner important significance of each site through using Definition 8.

$$Sig^{inner}(S_1, \mathbf{S}, D) = S_{\mathbf{S}-\{S_1\}}(D) - S_{\mathbf{S}}(d) = \frac{6}{10} - \frac{6}{10} = \frac{0}{10},$$
$$Sig^{inner}(S_2, \mathbf{S}, D) = S_{\mathbf{S}-\{S_2\}}(D) - S_{\mathbf{S}}(d) = \frac{6}{10} - \frac{6}{10} = \frac{0}{10},$$
$$Sig^{inner}(S_3, \mathbf{S}, D) = S_{\mathbf{S}-\{S_3\}}(D) - S_{\mathbf{S}}(d) = \frac{7}{10} - \frac{6}{10} = \frac{1}{10},$$
$$Sig^{inner}(S_4, \mathbf{S}, D) = S_{\mathbf{S}-\{S_4\}}(D) - S_{\mathbf{S}}(d) = \frac{7}{10} - \frac{6}{10} = \frac{1}{10},$$
$$Sig^{inner}(S_5, \mathbf{S}, D) = S_{\mathbf{S}-\{S_5\}}(D) - S_{\mathbf{S}}(d) = \frac{6}{10} - \frac{6}{10} = \frac{0}{10}.$$

Table 1b
The information system from site 2.

U	a_1	a_2
x_1	1	2
x_2	1	2
x_3	0	0
x_4	0	2
x_5	0	0
x_6	1	2
x_7	0	2
x_8	1	2
x_9	1	2
x_{10}	0	0

Table 1c
The information system from site 3.

U	a_2	a_3
x_1	2	1
x_2	2	1
x_3	0	0
x_4	0	0
x_5	0	0
x_6	1	1
x_7	1	1
x_8	2	1
x_9	2	1
x_{10}	0	0

Table 1d
The information system from site 4.

U	a_1	a_3
x_1	1	2
x_2	1	2
x_3	0	0
x_4	1	1
x_5	0	0
x_6	1	2
x_7	0	0
x_8	1	2
x_9	1	1
x_{10}	0	0

Table 1e
The information system from site 5.

U	a_1	a_2	a_3
x_1	1	2	2
x_2	1	2	2
x_3	0	1	1
x_4	0	1	2
x_5	0	1	1
x_6	0	1	2
x_7	0	1	1
x_8	1	2	2
x_9	1	2	2
x_{10}	0	1	1

According to the values of these inner significance measures, we select the site S_3 as first attribute in the heuristic algorithm (see Algorithm 1). Then, we select the second site through using Definition 9. Through calculating the outer significance measures of these sites, we have that

$$Sig^{outer}(S_1, S_3, D) = S_{\{S_3\}}(D) - S_{\{S_3 \cup S_1\}}(D) = \frac{8}{10} - \frac{8}{10} = \frac{0}{10},$$
$$Sig^{outer}(S_2, S_3, D) = S_{\{S_3\}}(D) - S_{\{S_3 \cup S_2\}}(D) = \frac{8}{10} - \frac{8}{10} = \frac{0}{10},$$
$$Sig^{outer}(S_4, S_3, D) = S_{\{S_3\}}(D) - S_{\{S_3 \cup S_4\}}(D) = \frac{8}{10} - \frac{6}{10} = \frac{2}{10},$$
$$Sig^{outer}(S_5, S_3, D) = S_{\{S_3\}}(D) - S_{\{S_3 \cup S_5\}}(D) = \frac{8}{10} - \frac{7}{10} = \frac{1}{10}.$$

In this situation, we select the fourth site as the selected one. Through calculating their pessimistic multigranulation lower approximation, we find

$$\underline{\{S_4, S_3\}}^P(D) = \underline{\mathbf{S}}^P(D) = \{\{x_1, x_2, x_8\}, \{x_3, x_5, x_{10}\}\}.$$

That is to say, $\{S_4, S_3\}$ is a reduct of these granular structures in the pessimistic multigranulation rough sets.

In addition, using the computational process above, we also can search an upper approximation reduct of the pessimistic multigranulation rough sets.

7. Conclusions

Multigranulation rough sets (MGRS) is one of desirable directions in rough set theory, in which lower/upper approximations are approximated by granular structures induced by multiple binary relations. It provides a new perspective for decision making analysis based on rough set theory. In this paper, we have first proposed a new multigranulation rough set model based on the decision strategy "Seeking common ground while eliminating differences" (SCED), called pessimistic rough set model. Moreover, we have discussed the relationship between optimistic multigranulation rough sets and pessimistic multigranulation rough sets. Finally, we have also developed a heuristic approach to find an attribute reduct from a decision table in the context of pessimistic multigranulation rough set model. These results largely enrich research scopes and applicable fields of rough set theory.

Following the development of multigranulation rough sets (MGRS), its future direction has four aspects: (1) model extension of multigranulation rough sets based on other binary relations; (2) information fusion based on multiple granular spaces; (3) information granule selection and granulation selection; and (4) applications of multigranulation rough sets. It is deserved to point out that the multigranulation rough set and the standard rough set can be combined for data mining and knowledge discovery from various real applications, such as multi-source information systems, data with high dimensions, and distributive information systems.

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